Journal of Statistical Physics, Vol. 113, Nos. 1/2, October 2003 (© 2003)

Noise Induced Dissipation in Lebesgue-Measure Preserving Maps on *d*-Dimensional Torus

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Received September 14, 2002; accepted April 9, 2003

We consider dissipative systems resulting from the Gaussian and *alpha*-stable noise perturbations of measure-preserving maps on the d dimensional torus. We study the dissipation time scale and its physical implications as the noise level ε vanishes. We show that nonergodic maps give rise to an $O(1/\varepsilon)$ dissipation time whereas ergodic toral automorphisms, including cat maps and their d-dimensional generalizations, have an $O(\ln(1/\varepsilon))$ dissipation time with a constant related to the minimal, *dimensionally averaged entropy* among the automorphism's irreducible blocks. Our approach reduces the calculation of the dissipation time to a nonlinear, arithmetic optimization problem which is solved asymptotically by means of some fundamental theorems in theories of convexity, Diophantine approximation and arithmetic progression. We show that the same asymptotic can be reproduced by degenerate noises as well as mere coarse-graining. We also discuss the implication of the dissipation time in kinematic dynamo.

KEY WORDS: Dissipation; noise; toral automorphisms; dynamo.

1. INTRODUCTION

Irreversibility and approach to equilibrium are fundamental problems in statistical mechanics and dynamical systems and its complete solution is still elusive (see, e.g., ref. 14). There are possibly many routes to irreversibility.

One view is that macroscopic systems are exceedingly difficult to isolate from their environments for a time comparable to their dynamical time scales. The noise as a result of interaction with environment may further trigger irreversibility, such as approach to equilibrium, in the

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0022-4715/03/1000-0335/0

systems. The initial uncertainty involved in preparing a physical system and the random perturbation due to measurements as well as Gibbs' coarse-graining procedure can all be viewed as certain noises. The point is that noises, intrinsic as a result of internal stochasticity or extrinsic as a result of random influence from surrounding environment, can induce effects that would be weak or absent without noise.

In this paper we investigate one such effect, called dissipation, for discrete time, conservative dynamical systems under the influence of noise. In particular we study the time scale, called the dissipation time, on which the dissipation as measured in L^p-norm, $1 < p < \infty$, has an *order one* effect even as the magnitude of noise vanishes. Clearly the dissipation time depends on the ergodic properties of the noiseless dynamics as well as the noise level.

The noisy dynamical system considered in this paper can be viewed as a discrete generalization of the dynamics of a passive scalar in a periodic, incompressible velocity field $\mathbf{v}$

$$\begin{aligned} d\mathbf{x}^{\varepsilon}(t) &= \mathbf{v}(\mathbf{x}^{\varepsilon}(t))\, dt + \sqrt{\varepsilon}\, d\mathbf{w}(t) \\ \nabla \cdot \mathbf{v}(\mathbf{x}) &= 0, \end{aligned} \tag{1}$$

where the standard Brownian motion $\mathbf{w}$ and the molecular diffusivity ε represent the stochastic perturbations as a result of random molecular collisions (see, e.g., refs. 13 and 8). The discrete-time dynamical system will be defined on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. The velocity field $\mathbf{v}$ will be replaced by arbitrary Lebesgue-measure preserving map F defined on $\mathbb{T}^d$ (periodicity condition).

In order to study the dynamics generated by F it is useful to consider its Koopman operator U_F defined by a composition $U_F f := f \circ F$, with f belonging to some Banach space of functions on $\mathbb{T}^d$. We will be mainly concerned with the standard Banach spaces $L^p(\mathbb{T}^d)$, $1 \leqslant p \leqslant \infty$, and their subspaces $L^p_0(\mathbb{T}^d)$ of functions with zero mean $\langle f \rangle = 0$, where $\langle f \rangle$ denotes the average of f w.r.t. the Lebesgue measure. In case of $L^1(\mathbb{T}^d)$ one can consider U_F as the Frobenius–Perron operator associated with F^{-1}.

In the time-discrete version we consider general α-stable noise operator $G_{\varepsilon,\alpha}\colon L^2_0(\mathbb{T}^d) \mapsto L^2_0(\mathbb{T}^d)$, with $\alpha \in (0, 1]$, defined by means of the Fourier transform of corresponding α-stable noise kernel $g_{\varepsilon,\alpha}$

$$G_{\varepsilon,\alpha} f(\mathbf{x}) = \int_{\mathbb{T}^d} g_{\varepsilon,\alpha}(\mathbf{x}-\mathbf{y})\, f(\mathbf{y})\, d\mathbf{y} = \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{-\varepsilon |\mathbf{k}|^{2\alpha}} \hat{f}(\mathbf{k})\, \mathbf{e}_{\mathbf{k}}(\mathbf{x}),$$

where

$$g_{\varepsilon,\alpha}(\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\varepsilon|\mathbf{k}|^{2\alpha}}\mathbf{e}_{\mathbf{k}}(\mathbf{x}),$$

with $\mathbf{e}_{\mathbf{k}}(\mathbf{x}) := e^{2\pi i\mathbf{k}\cdot\mathbf{x}}$, $\mathbf{k}\in\mathbb{Z}^d$. Here, just like in (1), $\varepsilon > 0$ represents the level of the noise. Putting $\alpha = 1$ one recovers standard heat kernel.

The operator $T_{\varepsilon,\alpha}$ on $L^2_0(\mathbb{T}^d)$ generating the noise-perturbed dynamical system considered in this paper is thus given by

$$T_{\varepsilon,\alpha} f := G_{\varepsilon,\alpha} U_F f = g_{\varepsilon,\alpha} * (f \circ F). \tag{2}$$

Simple computations yield

$$\|T^n_{\varepsilon,\alpha}\| = \|G_{\varepsilon,\alpha}U_F \cdots G_{\varepsilon,\alpha}U_F\| \leqslant \|G_{\varepsilon,\alpha}\|^n = e^{-\varepsilon n}. \tag{3}$$

Here and throughout the paper $\|\cdot\|$ denotes the standard L^2_0-norm or the corresponding operator norm (any other norm will be equipped with suitable index).

We define the dissipation time as the time on which the contraction (3) becomes of order one:

$$n_{\mathrm{diss}} := \min\{n\in\mathbb{Z}_+ : \|T^n_{\varepsilon,\alpha}\| < 1/e\}. \tag{4}$$

Hence the dissipation time is a function of ε, α as well as the underlying dynamics. The choice of the threshold e^{-1} in the definition is a convenient one and for the purpose of the paper can be any positive number less than one (see Proposition 1). The fact that for all $\varepsilon > 0$, $\|T^n_{\varepsilon,\alpha}\|$ is monotonically decreasing ensures that n_{diss} is well defined. By contrast, when $\varepsilon = 0$, the fine-grained Boltzmann–Gibbs entropy as well as L^p-norm of the initial state remains constant in the course of evolution. In other words this is a "dissipation" effect and hence the term "dissipation time." On the dissipation time scale the system is, in a sense, "half way" through its irreversible route to the equilibrium state. The dissipation time provides a measure of the instability of the dynamics w.r.t. the stochastic perturbations which result in the "aging" of the system toward the final state.

The main purpose of this paper is to investigate the asymptotics of the dissipation time as ε tends to zero. Due to the non-normality of the operator $T_{\varepsilon,\alpha}$, the dissipation time can not be determined from its spectral radius. Indeed, as we will see below, the operator $T_{\varepsilon,\alpha}$ corresponding to any ergodic toral automorphism F is quasi-nilpotent for any $\varepsilon > 0$ (thus the time scale estimated from the spectral radius is infinite) whereas the dissipation time is of the order $\ln(1/\varepsilon)$.

To briefly describe the main results we pause to note the following asymptotic notation. Given two sequences a_ε, b_ε indexed by the parameter $\varepsilon > 0$ we write

$$a_\varepsilon \lesssim b_\varepsilon, \qquad \text{if} \quad \limsup_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} < \infty$$

$$a_\varepsilon \approx b_\varepsilon, \qquad \text{if} \quad \lim_{\varepsilon \to 0} \frac{a_\varepsilon}{b_\varepsilon} = 1.$$

Moreover we write $a_\varepsilon \sim b_\varepsilon$, if both $a_\varepsilon \lesssim b_\varepsilon$ and $b_\varepsilon \lesssim a_\varepsilon$ hold simultaneously.

The first result is that the dissipation time $n_{\text{diss}} \sim 1/\varepsilon$ for nonergodic or, more general, non-weakly-mixing maps (cf. Theorem 1 and its corollaries, Section 2.3), which is also the longest possible time scale for dissipation in view of (3) (see also Lemma 1, Section 2.2). In other word, such systems are most stable w.r.t. stochastic perturbations.

The main aim, however, is to investigate the cases in which the dissipation is much faster due to rapid mixing in F. We show (Theorem 2, Section 2.3) that, for a toral automorphism F, $n_{\text{diss}} \sim \log(1/\varepsilon)$ if and only if the map F is ergodic (which in this case is also an Anosov diffeomorphism). In particular our results hold for all classical cat maps (hyperbolic automorphisms of 2-torus) and their d-dimensional generalizations. In addition, we provide a general lower bound for the constant of the logarithmic asymptotics (Theorem 2). We further show that the lower bound is achieved for *diagonalizable* automorphisms, namely

$$n_{\text{diss}} \approx (2\alpha \hat{h}(F))^{-1} \ln(1/\varepsilon) \tag{5}$$

where $\hat{h}(F)$ denotes the minimal, dimensionally averaged entropy among F's irreducible blocks (Theorem 3, Section 2.3). Dimensionally averaged entropy for each irreducible sub-block of the toral automorphism is the Kolmogorov–Sinai (KS) entropy per dimension of an irreducible factor of the whole map. Essentially all mixing Anosov diffeomorphisms should have the $\log 1/\varepsilon$ dissipation time but it is not immediately clear what the constant should be.

Our method involves solving asymptotically a quadratic arithmetic optimization (i.e., quadratic integer programming) problem by obtaining sharp upper and lower bounds using number theoretical tools including multidimensional Diophantine approximation theorems (Schmidt's subspace theorem), Minkowski's theorem on linear forms and Van der Waerden's theorem on arithmetic progressions. This is done in Section 3.

In Section 5 we show that the same result (5) holds when the noise is replaced by coarse-graining the initial and terminal states. This is reminiscent of the well-know results of statistical stability in the literature, namely, the Bernoulli systems are stable under the sufficiently small intrinsic random perturbation in the rough sense that the perturbed system is close to the direct product of the unperturbed one and some auxiliary viewer system (see refs. 13 and 21). In other words, for those systems, the process that results from small intrinsic random perturbation can be reproduced exactly by looking at the unperturbed system through a viewer that distorts randomly but slightly. In spite of the above the asymptotic (5) indicates the perturbed system is irreversibly far from the unperturbed one even on the relatively short dissipation time scale. From this perspective, such a system is statistically unstable.

In Section 4 we consider a class of highly degenerate noises and show that the same conclusions about the dissipation hold if the degenerate noises satisfy an additional generic condition.

In Section 6 we consider the relation between the dissipation time and some characteristic time scales relevant to kinematic dynamo. We show that fast dissipation generally inhibits dynamo action. When there is no fast dynamo action the noisy push-forward map dissipates the magnetic field energy on the dissipation time scale. However, the magnetic field energy can still grow to relatively large magnitude as inverse power-law of the small noise with the exponent proportional to the ratio of the logarithmic spectral radius of the toral automorphism to the minimal, dimensionally averaged entropy among the automorphism's irreducible blocks (cf. (65)).

The notion of dissipation time has a natural bearing on the problems of quantum chaos with noise. The family of symplectic toral automorphisms constitute important examples of quantizable chaotic dynamics on compact manifolds for which various quantization procedures have been intensively studied (see, for example, refs. 19 and 22). We will address the issue of decoherence time for quantized symplectic toral automorphisms with noise in a forthcoming paper.

The organization of the rest of the paper is as follows. In Section 2 we develop the general theory of dissipation time and its relation to the Boltzmann–Gibbs entropy. We also formulate the dissipation time calculation for total automorphisms as an arithmetic minimization problem and state the main results. In Appendix A we generalize the dissipation time asymptotic result to the affine transformations. The proofs of some elementary facts are presented in Appendix B for the sake of completeness.

2. DISSIPATION TIME

In its general form the dissipation time $n_{\text{diss}}(p)$ can be defined in terms of the norm $\|\cdot\|_{p,0}$ on the space $L_0^p(\mathbb{T}^d)$ w.r.t. a threshold $\eta \in (0, 1)$

$$n_{\text{diss}}(p, \eta) := \min\{n \in \mathbb{Z}_+ : \|T^n_{\varepsilon,\alpha}\|_{p,0} < \eta\}, \qquad 1 \leqslant p \leqslant \infty. \tag{6}$$

Note that $\|T^n_{\varepsilon,\alpha}\|_{p,0} = \|(T^*_{\varepsilon,\alpha})^n\|_{q,0}$, $p^{-1} + q^{-1} = 1$, $0 < p, q < \infty$ and thus $n_{\text{diss}}(p, \eta, F) = n_{\text{diss}}(q, \eta, F^{-1})$. First we show that the value of the threshold η in (6) does not affect the order of divergence of $n_{\text{diss}}(p, \eta)$, as ε tends to zero.

Proposition 1. For any $0 < \tilde{\eta}, \eta < 1$, $n_{\text{diss}}(p, \tilde{\eta}) \sim n_{\text{diss}}(p, \eta)$.

Proof. Assume $0 < \tilde{\eta} < \eta < 1$. Obviously $n_{\text{diss}}(p, \tilde{\eta}) \geqslant n_{\text{diss}}(p, \eta)$. On the other hand let k be a positive integer such that $\eta^k < \tilde{\eta}$. Then

$$\|T^{n_{\text{diss}}(p,\eta)}_{\varepsilon,\alpha}\|_{p,0} < \eta \Rightarrow \|T^{kn_{\text{diss}}(p)}_{\varepsilon,\alpha}\|_{p,0} < \eta^k < \tilde{\eta}. \tag{7}$$

Hence $kn_{\text{diss}}(p, \eta) \geqslant n_{\text{diss}}(p, \tilde{\eta})$, which implies $n_{\text{diss}}(p, \eta) \sim n_{\text{diss}}(p, \tilde{\eta})$. ∎

Following the argument of ref. 23 one can use the Riesz convexity theorem to establish also the asymptotic equivalence of the $n_{\text{diss}}(p)$, for all $1 < p < \infty$.

Proposition 2.

(i) For any $1 < q, p < \infty$, $n_{\text{diss}}(q) \sim n_{\text{diss}}(p)$.

(ii) For any $1 < p < \infty$, $n_{\text{diss}}(p) \lesssim n_{\text{diss}}(1)$ and $n_{\text{diss}}(p) \lesssim n_{\text{diss}}(\infty)$.

The details of the proof can be found in Appendix B.

Our particular choice of the exponent $p = 2$ and threshold $\eta = e^{-1}$ in (4) is computationally convenient and will be used throughout the paper. We will use the convention that $n_{\text{diss}}(p) = n_{\text{diss}}(p, e^{-1})$.

We say that operator $T_{\varepsilon,\alpha}$ or associated with it measure preserving map F has a *simple (slow)* dissipation time when $n_{\text{diss}} \sim 1/\varepsilon$ and that it has a *logarithmic (fast)* dissipation time when $n_{\text{diss}} \sim \ln(1/\varepsilon)$.

In the particular case of fast dissipation, with a logarithmic dissipation time, in order to estimate precisely the rate of dissipation, one needs to determine the value of the *dissipation rate constant* R_{diss}, defined as

$$R_{\text{diss}} := \lim_{\varepsilon \to 0} \frac{n_{\text{diss}}}{\ln(1/\varepsilon)}. \tag{8}$$

Similarly in case of simple dissipation time the dissipation rate constant can be defined as

$$R_{\text{diss}} := \lim_{\varepsilon \to 0} \varepsilon n_{\text{diss}}. \tag{9}$$

2.1. Dissipation Time and Boltzmann–Gibbs Entropy

In this section we briefly discuss the connection between dissipation time and Boltzmann–Gibbs entropy.

First we note that on the scale of n_{diss} the Boltzmann–Gibbs entropy approaches the maximal equilibrium value (i.e., 0) as can be seen from the following simple estimate.[16] Let us first restrict considerations to bounded initial states, i.e., $f \geqslant 0$, $f \in L^\infty$ and $\|f\|_1 = 1$. Let

$$\eta(u) = \begin{cases} -u \ln u, & u > 0 \\ 0, & u = 0 \end{cases}$$

and let $D_n = \{\mathbf{x} \in \mathbb{T}^d : 1 \leqslant T^n_{\varepsilon,\alpha} f\}$. On one hand, we have

$$\begin{aligned}
&\left| \int_{D_n} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x} \right| \\
&\qquad \leqslant \int_{D_n} \left| \int_1^{T^n_{\varepsilon,\alpha} f(\mathbf{x})} \frac{d\eta(u)}{du}\, du \right| d\mathbf{x} \\
&\qquad \leqslant \sup_{1 \leqslant u \leqslant \|T^n_{\varepsilon,\alpha} f\|_\infty} (1 + \ln u) \int_{D_n} |T^n_{\varepsilon,\alpha} f(\mathbf{x}) - 1|\, d\mathbf{x} \\
&\qquad \leqslant (1 + \ln \|T^n_{\varepsilon,\alpha} f\|_\infty)\, \|T^n_{\varepsilon,\alpha} f - 1\|_1 \\
&\qquad \leqslant (1 + \ln \|f\|_\infty)\, \|T^n_{\varepsilon,\alpha} f - 1\|_1.
\end{aligned} \tag{10}$$

On the other hand, we have

$$0 \geqslant \int_{\mathbb{T}^d} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x} \geqslant \int_{D_n} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x}.$$

In view of the inclusion relation: $L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$, we then obtain that for $n \gg n_{\text{diss}}$

$$\sup_{f \geqslant 0,\, \|f\|_\infty \leqslant c} \left| \int_{\mathbb{T}^d} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x} \right| \xrightarrow{\varepsilon \downarrow 0} 0, \quad \forall c > 0.$$

For unbounded initial states, we note that, by Young's inequality,

$$\|T^n_{\varepsilon,\alpha} f\|_\infty \leqslant \|T_{\varepsilon,\alpha} f\|_\infty \leqslant \|g_{\varepsilon,\alpha}\|_\infty \, \|f\|_1 = \|g_{\varepsilon,\alpha}\|_\infty$$

from which we have, instead of (10), the following estimate

$$\left|\int_{D_n} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x}\right| \leqslant (1 + \ln \|g_{\varepsilon,\alpha}\|_\infty)\, \|T^n_{\varepsilon,\alpha} f - 1\|_1$$

where

$$\ln \|g_{\varepsilon,\alpha}\|_\infty \sim \ln(1/\varepsilon).$$

Therefore for sufficiently fast diverging $n \gg n_{\mathrm{diss}}(1)$ such that

$$\ln(1/\varepsilon)\, \|T^n_{\varepsilon,\alpha}(f-1)\|_{1,0} \xrightarrow{\varepsilon \downarrow 0} 0 \tag{11}$$

one obtains

$$\sup_{f \geqslant 0,\, \|f\|_1 = 1} \left|\int_{\mathbb{T}^d} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x}\right| \xrightarrow{\varepsilon \downarrow 0} 0.$$

The condition (11) typically results in a slightly longer time scale than $n_{\mathrm{diss}}(1)$.

On the other hand, we can bound the L_1 distance between the probability density function f and the Lebesgue measure by their relative entropy via Csiszár's inequality[6]

$$\int_{\mathbb{T}^d} |f(\mathbf{x}) - g(\mathbf{x})|\, d\mathbf{x} \leqslant \sqrt{2 \int_{\mathbb{T}^d} f(\mathbf{x}) \ln(f(\mathbf{x})/g(\mathbf{x}))\, d\mathbf{x}}$$

with $g(\mathbf{x}) = 1$. We see immediately that the decay rate of

$$\sup_{f \geqslant 0,\, \|f\|_1 = 1} \left|\int_{\mathbb{T}^d} \eta(T^n_{\varepsilon,\alpha} f(\mathbf{x}))\, d\mathbf{x}\right|$$

provides an estimate for $n_{\mathrm{diss}}(1)$ and, consequently, for $n_{\mathrm{diss}}(p)$, $p \in (1, \infty)$.

2.2. Calculating the Dissipation Time

For greater generality and transparency of arguments we consider, in this section, a slightly more general family of operators $T_{\varepsilon,\alpha}$ defined, as previously, by the first equality in (2), but with arbitrary unitary or isometric (not necessary Koopman) operator U (and hence in these cases we drop the subscript F).

Lemma 1. For any isometric operator U, the dissipation time of $T_{\varepsilon,\alpha}$ satisfies following constraints

$$\|R(1;T_{\varepsilon,\alpha})\| \lesssim n_{\text{diss}} \lesssim 1/\varepsilon, \tag{12}$$

where $R(1;T_{\varepsilon,\alpha})$ denotes the resolvent of $T_{\varepsilon,\alpha}$ at 1.

Proof. In view of (4) and (3), for $n = n_{\text{diss}}$ one has

$$e^{-1} \leqslant \|T_{\varepsilon,\alpha}^{(n_{\text{diss}}-1)}\| \leqslant e^{-\varepsilon(n_{\text{diss}}-1)},$$

which clearly implies the second estimate of (12). In order to prove the other inequality we proceed as follows.

$$\|R(1;T_{\varepsilon,\alpha})\| = \left\|\sum_{n=0}^{\infty} T_{\varepsilon,\alpha}^{n}\right\| = \left\|\sum_{n=0}^{n_0-1} T_{\varepsilon,\alpha}^{n} + T_{\varepsilon,\alpha}^{n_0} \sum_{n=0}^{\infty} T_{\varepsilon,\alpha}^{n}\right\|$$
$$\leqslant \sum_{n=0}^{n_0-1} \|T_{\varepsilon,\alpha}^{n}\| + \|T_{\varepsilon,\alpha}^{n_0}\| \left\|\sum_{n=0}^{\infty} T_{\varepsilon,\alpha}^{n}\right\| \leqslant n_0 + \|T_{\varepsilon,\alpha}^{n_0}\|\,\|R(1;T_{\varepsilon,\alpha})\|.$$

Hence taking in the above inequality $n_0 = n_{\text{diss}}$ one gets

$$\|R(1;T_{\varepsilon,\alpha})\|\,(1-e^{-1}) \leqslant \|R(1;T_{\varepsilon,\alpha})\|\,(1-\|T_{\varepsilon,\alpha}^{n_{\text{diss}}}\|) \leqslant n_{\text{diss}},$$

which gives the first estimate of (12). ∎

The above lemma provides an absolute upper bound for dissipation time. Taking $F = I$ one easily finds that this bound is best possible in general. The lower bound is useful in the case when one can estimate from below the norm of the resolvent (see proof of Theorem 1).

Theorem 1. If U acting on $L_0^2(\mathbb{T}^d)$ possesses nonempty pure point spectrum and at least one of its eigenfunctions belongs to $H^{2\alpha}(\mathbb{T}^d)$, then $T_{\varepsilon,\alpha}$ has simple dissipation time.

Proof. In view of Lemma 1 it is enough to find a lower bound for the norm of the resolvent $R(1;T_{\varepsilon,\alpha})$. Let $h \in H^{2\alpha}$ be one of the eigenfunctions of U. Since U is isometric we have

$$Uh = e^{i\phi}h.$$

We first assume that $\phi = 0$. Since $1 \notin \sigma(T_{\varepsilon,\alpha})$, $I - T_{\varepsilon,\alpha}$ is a homeomorphism and hence

$$\|R(1;T_{\varepsilon,\alpha})\| = \sup_{f\in L_0^2} \frac{\|(I-T_{\varepsilon,\alpha})^{-1} f\|}{\|f\|} = \sup_{f\in L_0^2} \frac{\|f\|}{\|(I-T_{\varepsilon,\alpha})\,f\|} \geqslant \frac{\|h\|}{\|(I-T_{\varepsilon,\alpha})\,h\|}.$$

Now expressing h in the Fourier series we get

$$\|(I-T_{\varepsilon,\alpha})\, h\|^2 = \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} |\hat{h}(\mathbf{k})|^2 \, |1-e^{-\varepsilon |\mathbf{k}|^{2\alpha}}|^2 \leqslant \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} (\varepsilon \, |\hat{h}(\mathbf{k})| \, |\mathbf{k}|^{2\alpha})^2 \leqslant \varepsilon^2 \, \|h\|^2_{H^{2\alpha}}.$$

Hence

$$\|R(1; T_{\varepsilon,\alpha})\| \geqslant \frac{\|h\|}{\varepsilon \, \|h\|_{H^{2\alpha}}} =: \frac{C}{\varepsilon}.$$

Thus in view of (12) and above calculations

$$1/\varepsilon \lesssim \|R(1; T_{\varepsilon,\alpha})\| \lesssim n_{\text{diss}} \lesssim 1/\varepsilon,$$

which ends the proof in the case $\phi = 0$.

If $\phi \neq 0$, we put $\hat{U} = e^{-i\phi} U$, which implies $\hat{U} h = h$.

The proof is completed by applying the above reasoning to operator $\hat{T}_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} \hat{U}$ and observing that the dissipation times for $T_{\varepsilon,\alpha}$ and $\hat{T}_{\varepsilon,\alpha}$ are identical. ▮

When U is a Koopman operator associated with a map F, then the property that U considered on $L^2_0(\mathbb{T}^d)$ possesses nonempty pure point spectrum is equivalent to the fact that F is not weakly mixing (see ref. 5). Thus we have

Corollary 1. If F is not weakly mixing and its Koopman operator possesses $H^{2\alpha}$ eigenfunction in $L^2_0(\mathbb{T}^d)$, then $T_{\varepsilon,\alpha}$ has simple dissipation time.

Another immediate consequence is

Corollary 2. If F is not ergodic and its nontrivial invariant measure possesses $H^{2\alpha}$ density function, then $T_{\varepsilon,\alpha}$ has simple dissipation time.

A typical example of ergodic but not weakly mixing transformations for which the above corollary applies is the family of "irrational" shifts on $\mathbb{T}^d$, i.e., maps $F\mathbf{x} = \mathbf{x} + \mathbf{c}$ on $\mathbb{T}^d$, where $\mathbf{c} = (c_1, ..., c_d)$ is a constant vector such that the numbers $1, c_1, ..., c_d$ are linearly independent over rationals. More general and less trivial examples of ergodic maps giving rise to a simple dissipation time will be discussed in Appendix A (cf. Remark 1).

In general the problem of computing the dissipation time is rather complicated. In some cases it can be reformulated as an asymptotic optimization problem. To see it, one can represent the action of a given unitary operator U in the Fourier basis

$$U\mathbf{e}_{\mathbf{k}} = \sum_{0 \neq \mathbf{k}' \in \mathbb{Z}^d} u_{\mathbf{k},\mathbf{k}'} \mathbf{e}_{\mathbf{k}'},$$

where for each $\mathbf{k}$

$$\sum_{0 \neq \mathbf{k}' \in \mathbb{Z}^d} |u_{\mathbf{k},\mathbf{k}'}|^2 = 1. \tag{13}$$

Next we introduce the notation

$$\mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n) = \sum_{0 \neq \mathbf{k}_1,\dots,\mathbf{k}_{n-1} \in \mathbb{Z}^d} u_{\mathbf{k}_0,\mathbf{k}_1} \cdots u_{\mathbf{k}_{n-1},\mathbf{k}_n} e^{-\varepsilon \sum_{l=1}^{n} |\mathbf{k}_l|^{2\alpha}}$$
$$\mathscr{S}_n(\mathbf{k}_n) = \{\mathbf{k}_0 \in \mathbb{Z}^d \setminus \{0\} : \mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n) \neq 0\}.$$

Then for any $f \in L_0^2(\mathbb{T}^d)$ we have

$$\begin{aligned}
\|T^n_{\varepsilon,\alpha} f\|^2 &= \Big\| \sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} \hat{f}(\mathbf{k}_0)\, T^n_{\varepsilon,\alpha} \mathbf{e}_{\mathbf{k}_0} \Big\|^2 = \Big\| \sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} \hat{f}(\mathbf{k}_0) \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n)\, \mathbf{e}_{\mathbf{k}_n} \Big\|^2 \\
&= \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \Big| \sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} \hat{f}(\mathbf{k}_0)\, \mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n) \Big|^2 \\
&= \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \Big| \sum_{\mathbf{k}_0 \in \mathscr{S}_n(\mathbf{k}_n)} \hat{f}(\mathbf{k}_0)\, \mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n) \Big|^2.
\end{aligned} \tag{14}$$

The following general upper bound for $\|T^n_{\varepsilon,\alpha} f\|$ holds.

Lemma 2. For any $f \in L_0^2(\mathbb{T}^d)$,

$$\|T^n_{\varepsilon,\alpha} f\|^2 \leqslant \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \sum_{\mathbf{k}_0 \in \mathscr{S}_n(\mathbf{k}_n)} |\hat{f}(\mathbf{k}_0)|^2 \sum_{\mathbf{k}_0 \in \mathscr{S}_n(\mathbf{k}_n)} |\mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n)|^2. \tag{15}$$

For the proof we refer the reader to Appendix B.

When $u_{\mathbf{k},\mathbf{k}'}$ is a Kronecker's delta function

$$u_{\mathbf{k},\mathbf{k}'} = \delta_{A\mathbf{k},\mathbf{k}'}, \tag{16}$$

where $A: \mathbb{Z}^d \mapsto \mathbb{Z}^d$ is a linear surjective map, the upper bound (15) can be used to obtain an identity for $\|T^n_{\varepsilon,\alpha}\|$. First observe that

$$\mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n) = e^{-\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}_0|^{2\alpha}} \delta_{A^n \mathbf{k}_0, \mathbf{k}_n}$$

and hence (15) becomes

$$\|T^n_{\varepsilon,\alpha} f\|^2 \leqslant \sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} |\hat{f}(\mathbf{k}_0)|^2 \, e^{-2\varepsilon \sum_{l=1}^n |A^l \mathbf{k}_0|^{2\alpha}} \leqslant \|f\|^2 \max_{0 \neq \mathbf{k} \in \mathbb{Z}^d} e^{-2\varepsilon \sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha}}.$$

On the other hand for any nonzero $\mathbf{k} \in \mathbb{Z}^d$, one can take in (14) $f = \mathbf{e}_\mathbf{k}$ and get

$$\|T^n_{\varepsilon,\alpha} f\|^2 = e^{-2\varepsilon \sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha}}$$

and therefore

$$\|T^n_{\varepsilon,\alpha}\| = \max_{0 \neq \mathbf{k} \in \mathbb{Z}^d} e^{-\varepsilon \sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha}} = e^{-\varepsilon \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha}}. \tag{17}$$

Let us now determine the class of maps F such that the corresponding Koopman operator U_F satisfies (16). The relation (16) implies

$$U_F \mathbf{e}_\mathbf{k} = \mathbf{e}_{A\mathbf{k}} = e^{2\pi i \langle A\mathbf{k}, \mathbf{x} \rangle}.$$

On the other hand

$$U_F \mathbf{e}_\mathbf{k}(\mathbf{x}) = \mathbf{e}_\mathbf{k}(F\mathbf{x}) = e^{2\pi i \langle \mathbf{k}, F\mathbf{x} \rangle}.$$

Thus

$$\langle \mathbf{k}, F\mathbf{x} \rangle = \langle A\mathbf{k}, \mathbf{x} \rangle \bmod 1, \qquad \forall \mathbf{x} \in \mathbb{R}^d, \quad \mathbf{k} \in \mathbb{Z}^d,$$

that is, A is linear and $A^\dagger$ equals the lifting of F from $\mathbb{T}^d$ onto $\mathbb{R}^d$. Moreover, the matrix A has integer entries and determinant equal to ± 1, i.e., A (and F) is a toral automorphism. Hence, for toral automorphisms, the calculation of the dissipation time reduces to the following nonlinear, asymptotic (large n) arithmetic minimization problem

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha}. \tag{18}$$

We will show in Section 3 that for any ergodic toral automorphism this minimum value grows geometrically in n with the base related to the dimensionally-averaged KS-entropy of the total automorphism.

2.3. Dissipation Time of Toral Automorphisms

It is well known (see ref. 1) that (the lifting map corresponding to) any toral homeomorphism $H: \mathbb{T}^d \mapsto \mathbb{T}^d$ can be decomposed into three parts

$H = L + P + c$, where L, the linear part, is an element of $SL(d, \mathbb{Z})$—the set of all matrices with integer entries and determinant equal to ± 1, P is periodic, i.e., $P(\mathbf{x}+\mathbf{v}) = P(\mathbf{x})$ for any $\mathbf{v} \in \mathbb{Z}^d$, and c is a constant shift vector.

Every algebraic and measurable automorphism of the torus is continuous. Each continuous toral automorphism is a homeomorphism with zero periodic and constant parts and hence can be identified with an element of $SL(d, \mathbb{Z})$. And vice versa, each element of $SL(d, \mathbb{Z})$ uniquely determines a measurable, algebraic toral automorphism. Thus from now on the term *toral automorphism* will simply be reserved for elements of $SL(d, \mathbb{Z})$. We recall here that all Anosov diffeomorphisms on $\mathbb{T}^d$ are topologically conjugate to the toral automorphisms (refs. 9 and 18).

Below we summarize some ergodic properties of toral automorphisms (cf. ref. 11, p. 160, refs. 3 and 12).

Proposition 3. Let F be a toral automorphism. The following statements are equivalent

(a) no root of unity is an eigenvalue of F.

(b) F is ergodic.

(c) F is mixing.

(d) F is a K-system.

(e) F is a Bernoulli system.

In the sequel we will use the following result (cf. ref. 26).

Proposition 4. The entropy $h(F)$ of any toral endomorphism F is computed by the formula

$$h(F) = \sum_{|\lambda_j| \geqslant 1} \ln |\lambda_j|, \tag{19}$$

where λ_j denote the eigenvalues of A.

From the formula (19) one immediately sees that a toral automorphism has zero entropy if and only if all its eigenvalues are of modulus 1. In fact much stronger result holds.

Proposition 5. A toral automorphism has zero entropy if and only if all its eigenvalues are roots of unity. In particular all ergodic toral automorphisms have positive entropy.

Given any toral automorphism F we denote by P its characteristic polynomial and by $\{P_1,..., P_s\}$ the complete set of its distinct irreducible (over $\mathbb{Q}$) factors. Let d_j denote the degree of polynomial P_j and h_j the KS-entropy of any toral automorphism with the characteristic polynomial P_j. For each P_j we define its dimensionally averaged KS-entropy as

$$\hat{h}_j = \frac{h_j}{d_j}. \tag{20}$$

For the whole matrix F we define its minimal dimensionally averaged entropy (denoted $\hat{h}(F)$) as

$$\hat{h}(F) = \min_{j=1,\dots,s} \hat{h}_j.$$

Now we state two main theorems of the present paper.

Theorem 2. Let F be any toral automorphism, U_F the Koopman operator associated with F, $G_{\varepsilon,\alpha}$ α-stable noise operator and $T_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} U_F$. Then

(i) $T_{\varepsilon,\alpha}$ has simple dissipation time if and only if F is not ergodic.

(ii) $T_{\varepsilon,\alpha}$ has logarithmic dissipation time if and only if F is ergodic.

(iii) If $T_{\varepsilon,\alpha}$ has logarithmic dissipation time then the dissipation rate constant satisfies the following constraint

$$\frac{1}{2\alpha\hat{h}(F)} \leqslant R_{\text{diss}} \leqslant \frac{1}{2\alpha\tilde{h}(F)},$$

where $\tilde{h}(F)$ is a positive constant satisfying $\tilde{h}(F) \leqslant \hat{h}(F)$.

Part (i) of the above theorem follows immediately from Theorem 1. For details of a simple proof we refer to Appendix B.

The natural question arises, whether the lower bound for the dissipation rate constant given in the above theorem is best possible. The next theorem and its corollary provides a strong argument in favor of this conjecture.

Theorem 3. If F is ergodic and diagonalizable then

$$n_{\text{diss}} \approx \frac{1}{2\alpha\hat{h}(F)} \ln(1/\varepsilon).$$

That is, the dissipation rate constant of $T_{\varepsilon,\alpha}$ is given by

$$R_{\text{diss}} = \frac{1}{2\alpha\hat{h}(F)}.$$

The proof of parts (ii) and (iii) of Theorem 2 and of Theorem 3 constitute the most important part of this work and will be presented in Section 3.3 after necessary tools are developed.

We end this section with the the results for two and three dimensional tori. Ergodicity of two dimensional toral automorphisms is equivalent to hyperbolicity. Two dimensional hyperbolic toral automorphisms are often referred to as the *cat maps*.

Using Corollary 4 and applying Theorem 3 to two and three dimensions one gets the following

Corollary 3. Let F be any ergodic, two or three dimensional toral automorphism. Then

$$n_{\text{diss}} \approx \frac{1}{2\alpha\hat{h}(F)} \ln(1/\varepsilon).$$

3. ASYMPTOTIC ARITHMETIC MINIMIZATION PROBLEM

In this section we find the asymptotics, as n goes to infinity, of the following quadratic arithmetic minimization problem

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}, \tag{21}$$

where $A \in SL(d, \mathbb{Z})$. When A is not ergodic the asymptotics of (21) is of the order $O(n)$. For the rest of the paper we will consider only the ergodic case. For $d = 2$ the problem (21) can be solved easily as follows. Consider first the case that A is symmetric and $\alpha = 1$. From $\det(A) = 1$ we see that eigenvalues are λ, λ^{-1} with $|\lambda| > 1$. We have

$$\begin{aligned}
&\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{2n+1} |A^l \mathbf{k}|^2 \\
&\quad = \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=-n}^{n} |A^l \mathbf{k}|^2 \\
&\quad = \min_{0 \neq \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \in \mathbb{Z}^d} \left(|\mathbf{k}|^2 + \sum_{l=1}^{n} |\lambda|^{2l} |\mathbf{k}_1|^2 + |\lambda|^{-2l} |\mathbf{k}_2|^2 + \sum_{l=1}^{n} |\lambda|^{-2l} |\mathbf{k}_1|^2 + |\lambda|^{2l} |\mathbf{k}_2|^2 \right) \\
&\quad = \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=-n}^{n} |\lambda|^{2l} |\mathbf{k}|^2 = \sum_{l=-n}^{n} |\lambda|^{2l}.
\end{aligned}$$

Hence there exist constants C_1 and C_2 such that

$$C_1 e^{h(A)\,n} \leqslant \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant C_2 e^{h(A)\,n}$$

where $h(A)$ denotes the KS-entropy of A. The estimates for the general case of non-symmetric A and $\alpha \neq 1$ are similar.

In higher dimensions, the solution to (21) is much more involved because of the presence of different eigenvalues with absolute values bigger than one. We have the following general estimate

Theorem 4. Let $A \in SL(d, \mathbb{Z})$ be ergodic. There exist constants C_1 and C_2 such that for any $0 < \delta < 1$ and sufficiently large n

$$C_1 e^{(1-\delta)\, 2\alpha \tilde{h}(A)\, n} \leqslant \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant C_2 n e^{2\alpha \hat{h}(A)\, n} \tag{22}$$

where as before $\hat{h}(A)$ denotes minimal dimensionally averaged entropy of A and $\tilde{h}(A)$ denotes a constant satisfying $0 < \tilde{h}(A) \leqslant \hat{h}(A)$, with equality achieved for all diagonalizable matrices A.

The question whether the equality $\tilde{h}(A) = \hat{h}(A)$ holds for all ergodic matrices remains open.

The proof of the theorem relies on nontrivial use of three number-theoretical results stated below.

I. Minkowski's Theorem on Linear Forms. Let $L_1,\dots, L_d$ be linearly independent linear forms on $\mathbb{R}^d$ which are real or occur in conjugate complex pairs. Suppose $a_1, a_2,\dots, a_d$ are real positive numbers satisfying $a_1 a_2 \cdots a_d = 1$ and $a_i = a_j$, whenever $L_i = \bar{L}_j$. Then there exists a nonzero integer vector $\mathbf{k} \in \mathbb{Z}^d$ such that for every $j = 1,\dots, d$,

$$|L_j \mathbf{k}| \leqslant D a_j, \tag{23}$$

where $D = |\det[L_1,\dots, L_d]|^{1/d}$.

Minkowski's Theorem on linear forms will be used to obtain a sharp upper bound on the asymptotic solution of the arithmetic minimization problem. The proof of the above theorem and its generalization to arbitrary lattices can be found in ref. 20 (Chap. VI).

II. Schmidt's Subspace Theorem. Let $L_1,\dots, L_d$ be linearly independent linear forms on $\mathbb{R}^d$ with real or complex algebraic coefficients.

Given $\delta > 0$, there are finitely many proper rational subspaces of $\mathbb{R}^d$ such that every nonzero integer vector $\mathbf{k}$ with

$$\prod_{j=1}^{d} |L_j \mathbf{k}| < |\mathbf{k}|^{-\delta} \tag{24}$$

lies in one of these subspaces.

Schmidt's Subspace Theorem will be used in conjunction with Van der Waerden's Theorem on arithmetic progressions (see below) to obtain a sharp lower bound for the asymptotic solution of the arithmetic minimization problem. The proof of Schmidt's Subspace Theorem can be found in ref. 24 (Theorem 1F, p. 153).

Definition 1. For a given set of linear forms and for fixed $\delta > 0$, the smallest collection of proper rational subspaces of $\mathbb{R}^d$ which contain all nonzero integer vectors satisfying (24), is called the exceptional set and denoted by E_δ.

A main difficulty to be resolved in using Schmidt's Subspace Theorem is to show that the minimizer of either the original problem (21) or an equivalent problem does not lie in the respective exceptional set which is in general unknown. We will pursue the latter route by using Van der Waerden's Theorem on arithmetic progressions to show that one can always construct an equivalent minimization problem whose minimizer is guaranteed to lie outside the corresponding exceptional set. To this end we note that Schmidt's Subspace Theorem is true when the standard lattice $\mathbb{Z}^d$ is replaced by any other rational lattice, that is any lattice of the form $\Lambda = Q(\mathbb{Z}^d)$ where $Q \in GL(d, \mathbb{Q})$. Schmidt's subspace theorem can be generalized to this situation by considering the set of new forms $\tilde{L}_j = L_j Q$. The fact that $Q \in GL(d, \mathbb{Q})$ implies immediately that $\tilde{L}_j$ are still linearly independent forms on $\mathbb{R}^d$ with real or complex algebraic coefficients.

III. Van der Waerden's Theorem on Arithmetic Progressions. Let k and d be two arbitrary natural numbers. Then there exists a natural number $n_*(k, d)$ such that, if an arbitrary segment of length $n \geqslant n_*$ of the sequence of natural numbers is divided in any manner into k (finite) subsequences, then an arithmetic progression of length d appears in at least one of these subsequences.

The original proof was published in ref. 25; Lukomskaya's simplification can be found in ref. 17.

Before presenting the proof of our main results we state a number of technical facts concerning the structure of toral automorphisms.

3.1. Algebraic Structure of Toral Automorphisms

In this section we denote by $GL(d, \mathbb{Q})$ the group of nonsingular $d \times d$ matrices with rational entries or the group of linear operators on Euclidean space $\mathbb{R}^d$, which are represented in standard basis by such matrices. We generally use the same symbol to denote both operator and its matrix.

In the sequel a vector $x \in \mathbb{R}^d$ will be called an integer (or integral) vector if all its components are integers, and similarly a rational, an algebraic vector if all its components are rational or respectively algebraic numbers. The term *rational subspace of* $\mathbb{R}^d$ will then refer to a linear subspace of $\mathbb{R}^d$ spanned by rational vectors (cf. ref. 24, p. 113).

Definition 2. $A \in GL(d, \mathbb{Q})$ is called irreducible (over $\mathbb{Q}$) if its characteristic polynomial is irreducible in $\mathbb{Q}[x]$.

Lemma 3. The following statements about a matrix $A \in GL(d, \mathbb{Q})$ are equivalent.

(a) A is irreducible.

(b) A does not possess any proper rational A-invariant subspaces of $\mathbb{R}^d$.

(c) No rational proper subspace of $\mathbb{R}^d$ is contained in any proper A-invariant subspace of $\mathbb{R}^d$.

(d) For any nonzero $\mathbf{q} \in \mathbb{Q}^d$ and any arithmetic progression of integer numbers $n_1,..., n_d$, the set $\{A^{n_1}\mathbf{q}, A^{n_2}\mathbf{q},..., A^{n_d}\mathbf{q}\}$ forms a basis of $\mathbb{R}^d$.

(e) $A^\dagger$ is irreducible.

(f) No nonzero $\mathbf{q} \in \mathbb{Q}^d$ is orthogonal to any proper A-invariant subspace of $\mathbb{R}^d$.

(g) No proper A-invariant subspace of $\mathbb{R}^d$ is contained in any proper rational subspace of $\mathbb{R}^d$.

Definition 3. We say that operator $A \in GL(d, \mathbb{Q})$ is completely decomposable over $\mathbb{Q}$ if there exists a rational basis of $\mathbb{R}^d$ in which A admits the following block diagonal form

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_r \end{bmatrix}, \tag{25}$$

where for each $j = 1,..., r \leqslant d$, $A_j \in GL(d_j, \mathbb{Q})$ is irreducible and $\sum_{j=1}^{r} d_j = d$.

In general, any matrix $A \in GL(d, \mathbb{Q})$ admits a rational block diagonal representation $[A_j]_{j=1,\dots,r}$. The smallest rational blocks to which A can be decomposed are called elementary divisor blocks. The characteristic polynomial corresponding to any elementary divisor block is of the form p^m, where p is an irreducible (over $\mathbb{Q}$) polynomial (see, e.g., ref. 7). Although elementary divisor blocks cannot be decomposed over $\mathbb{Q}$ into smaller invariant blocks, some elementary divisor blocks may not be irreducible. This happens iff $m > 1$ iff A is not completely decomposable over $\mathbb{Q}$. One has the following elementary fact (see Appendix B for a proof).

Proposition 6. $A \in GL(d, \mathbb{Q})$ is completely decomposable over $\mathbb{Q}$ iff A is diagonalizable.

However, even if $A \in GL(d, \mathbb{Q})$ is not completely decomposable, each elementary divisor block of A can be uniquely represented (in a rational basis) in the following block upper triangular form

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{26}$$

where B is the unique rational irreducible sub-block associated with A-invariant rational subspace of that elementary divisor and C, D denote some rational matrices.

Proposition 7. All the eigenvalues of an irreducible matrix $A \in GL(d, \mathbb{Q})$ are distinct (complex) algebraic numbers. In particular all irreducible matrices are diagonalizable.

The proofs of the above propositions can be found in Appendix B.

Finally we note that since the leading coefficient and constant term of a characteristic polynomial of any toral automorphism are equal to 1, the only possible rational eigenvalues of such map are ± 1 or $\pm i$. The latter fact implies that ergodic toral automorphisms do not possesses rational eigenvalues. Thus we have the following

Corollary 4. Let F be an ergodic, two or three dimensional toral automorphism. Then F is irreducible (and hence diagonalizable).

3.2. Proof of Theorem 4

This section is entirely devoted to the proof of Theorem 4.

Let $[A_j]_{j=1,\dots,r}$ be a rational block-diagonal decomposition of A into elementary divisor blocks. Since $A \in SL(d, \mathbb{Z})$, there exist a transition matrix $Q \in SL(d, \mathbb{Q})$ such that for every $l \in \mathbb{Z}$,

$$A^l = Q^{-1}([A_j])^l Q \tag{27}$$

and moreover each elementary divisor block $[A]_j$ is represented in its block upper triangular form (26).

The matrix Q defines a new lattice $\Lambda = Q(\mathbb{Z}^d)$ and acts bijectively between this lattice and the standard lattice $\mathbb{Z}^d$. Hence

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} = \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |Q^{-1}([A_j])^l Q\mathbf{k}|^{2\alpha} = \min_{0 \neq \mathbf{q} \in \Lambda} \sum_{l=1}^{n} |Q^{-1}([A_j])^l \mathbf{q}|^{2\alpha}.$$

Moreover

$$\|Q\|^{-2\alpha} |([A_j])^l \mathbf{q}|^{2\alpha} \leqslant |Q^{-1}([A_j])^l \mathbf{q}|^{2\alpha} \leqslant \|Q^{-1}\|^{2\alpha} |([A_j])^l \mathbf{q}|^{2\alpha}, \qquad \forall l, j, \alpha.$$

Now we decompose Λ into the direct sum of lower dimensional sublattices Λ_j corresponding to invariant blocks $[A_j]$. So that

$$\min_{0 \neq \mathbf{q} \in \Lambda} \sum_{l=1}^{n} |([A_j])^l \mathbf{q}|^{2\alpha} = \min_{j \in \{1,\dots,r\}} \min_{0 \neq \mathbf{q} \in \Lambda_j} \sum_{l=1}^{n} |(A_j)^l \mathbf{q}|^{2\alpha}. \tag{28}$$

Thus, without loss of generality, we may specialize to the case that A is already indecomposable over $\mathbb{Q}$, i.e., A does not possesses any proper elementary divisor blocks. To simplify the notation we will work with the standard lattice $\Lambda = \mathbb{Z}^d$. According to the remarks following the statements of Minkowski's and Schmidt's Theorems the proof can be easily adapted for any rational lattice $\Lambda = Q(\mathbb{Z}^d)$.

Since the technique of the proof differs depending on diagonalizability of A we consider two cases:

3.2.1. Diagonalizable Case

Here we concentrate on the case when A is diagonalizable and hence due to its in-decomposability irreducible (cf. Proposition 6).

We denote by λ_j $(j = 1,\dots, d)$ the eigenvalues of A. Following Proposition 7 we note that λ_j are distinct (possibly complex) algebraic numbers and hence there exists a basis (of $\mathbb{C}^d$) $\{\mathbf{v}_j\}_{j=1,\dots,d}$ composed of normalized algebraic eigenvectors corresponding to eigenvalues λ_j.

We denote by $[P_j]_{j=1}^d$ the projections on $[\mathbf{v}_j]$, and by $[L_j]$ the corresponding linear forms. It is easy to check that $[L_j]$ are given, in the Riesz identification, by the eigenvectors $[\mathbf{u}_j]$ of the matrix $A^\dagger$ which are co-orthogonal to $[\mathbf{v}_j]$, i.e., $\langle \mathbf{u}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$. $[\mathbf{u}_j]$ and $[\mathbf{v}_j]$ are real or occur in complex conjugate pairs. We have

$$\mathbf{x} = \sum_{j=1}^d P_j \mathbf{x} = \sum_{j=1}^d (L_j \mathbf{x})\, \mathbf{v}_j = \sum_{j=1}^d \langle \mathbf{x}, \mathbf{u}_j \rangle\, \mathbf{v}_j, \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

The equivalence between any two norms in a finite dimensional vector space, implies the existence of absolute constants C_1, C_2 such that

$$C_1 \sum_{j=1}^d |P_j \mathbf{x}|^2 \leqslant |\mathbf{x}|^2 \leqslant C_2 \sum_{j=1}^d |P_j \mathbf{x}|^2.$$

Using the above inequalities, the monotonicity of a map $\mathbf{x} \mapsto \mathbf{x}^\alpha$ and an obvious inequality $(a+b)^\alpha \leqslant a^\alpha + b^\alpha$, which holds for all positive a, b, and $\alpha \in (0, 1]$ one obtains the following estimates

$$\begin{aligned}\sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha} &\leqslant \sum_{l=1}^n \left(C_2 \sum_{j=1}^d |P_j A^l \mathbf{k}|^2 \right)^\alpha = C_2^\alpha \sum_{l=1}^n \left(\sum_{j=1}^d |\lambda_j|^{2l}\, |P_j \mathbf{k}|^2 \right)^\alpha \\ &\leqslant C_2^\alpha \sum_{l=1}^n \sum_{j=1}^d |\lambda_j|^{2\alpha l}\, |P_j \mathbf{k}|^{2\alpha} = C_2^\alpha \sum_{j=1}^d \left(\sum_{l=1}^n |\lambda_j|^{2\alpha l} \right) |P_j \mathbf{k}|^{2\alpha}\end{aligned}$$

and on the other hand

$$\begin{aligned}\sum_{l=1}^n |A^l \mathbf{k}|^{2\alpha} &\geqslant \left(\sum_{l=1}^n |A^l \mathbf{k}|^2 \right)^\alpha \geqslant \left(\sum_{l=1}^n C_1 \sum_{j=1}^d |P_j A^l \mathbf{k}|^2 \right)^\alpha \\ &= C_1^\alpha \left(\sum_{l=1}^n \sum_{j=1}^d |\lambda_j|^{2l}\, |P_j \mathbf{k}|^2 \right)^\alpha = C_1^\alpha \left(\sum_{j=1}^d \left(\sum_{l=1}^n |\lambda_j|^{2l} \right) |P_j \mathbf{k}|^2 \right)^\alpha.\end{aligned}$$

Now we introduce some notation

$$\hat{\lambda}_j := \max\{1, |\lambda_j|\}, \tag{29}$$

$$\hat{\lambda}_{\text{geo}} := \left(\prod_{j=1}^d \hat{\lambda}_j \right)^{1/d}. \tag{30}$$

One can easily observe that there exists a constant C such that

$$C \hat{\lambda}_j^{2\alpha n} \leqslant \sum_{l=1}^n |\lambda_j|^{2\alpha l} \leqslant n \hat{\lambda}_j^{2\alpha n}.$$

In the sequel we do not distinguish between particular values of constants appearing in computations. The symbols $C_1, C_2,...$ are used to denote any generic constants independent of n.

The normalization condition $|\mathbf{v}_j| = 1$ implies the following relation

$$|P_j \mathbf{x}| = |L_j \mathbf{x}|. \tag{31}$$

Combining the above estimates one gets the following general bounds

$$C_1 \left(\sum_{j=1}^{d} \hat{\lambda}_j^{2n} |L_j \mathbf{k}|^2 \right)^{\alpha} \leqslant \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant C_2 n \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{k}|^{2\alpha}. \tag{32}$$

Therefore in order to estimate (21) it suffices, essentially, to estimate

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{k}|^{2\alpha}. \tag{33}$$

We denote by $\mathbf{z}_n$ the sequence of minimizers, i.e., nonzero integral vectors solving (33).

Upper Bound. For the upper bound we assign to the set of linear forms L_j the set $\mathscr{A}$ composed of all real vectors $\mathbf{a} = (a_1,..., a_d)$ satisfying the conditions $a_j > 0$, for $j = 1,..., d$ and $a_i = a_j$ whenever $L_i = \bar{L}_j$ and

$$\prod_{j=1}^{d} a_j = 1. \tag{34}$$

From Minkowski's theorem on linear forms, we know that for any $\mathbf{a} \in \mathscr{A}$, there exists nonzero integral vector $\mathbf{k}_\mathbf{a}$ satisfying $|L_j \mathbf{k}_\mathbf{a}| \leqslant D a_j$, $j = 1,..., d$, where $D = |\det[L_1,..., L_d]|^{1/d}$.

Thus

$$\sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{k}_\mathbf{a}|^{2\alpha} \leqslant D \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} a_j^{2\alpha}. \tag{35}$$

The minimizing property of $\mathbf{z}_n$ implies that for any $\mathbf{a} \in \mathscr{A}$,

$$\sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{z}_n|^{2\alpha} \leqslant \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{k}_\mathbf{a}|^{2\alpha}. \tag{36}$$

Thus combining (35) and (36), and applying the Lagrange multipliers minimization with the constraint (34) (and using the fact that $\hat{\lambda}_i = \hat{\lambda}_j$ whenever $L_i = \bar{L}_j$), we get

$$\sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{z}_n|^{2\alpha} \leqslant D \min_{\mathbf{a} \in \mathscr{A}} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} a_j^{2\alpha} = dD \left(\prod_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} \right)^{1/d} = dD\, \hat{\lambda}_{\mathrm{geo}}^{2\alpha n}. \quad (37)$$

Thus the following upper bound holds

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant C_2 n \hat{\lambda}_{\mathrm{geo}}^{2\alpha n}. \quad (38)$$

Lower Bound. Let m denote an arbitrary natural number. Using the fact that A acts bijectively on $\mathbb{Z}^d$ we can restate the minimization problem (33) in the following form

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{k}|^{2\alpha} = \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j A^{-m} A^m \mathbf{k}|^{2\alpha} \quad (39)$$

$$= \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |\lambda_j|^{-2\alpha m} |L_j A^m \mathbf{k}|^{2\alpha}. \quad (40)$$

That is

$$\sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{z}_n|^{2\alpha} = \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |\lambda_j|^{-2\alpha m} |L_j A^m \mathbf{z}_n|^{2\alpha}. \quad (41)$$

We choose arbitrary $\delta > 0$ and consider the exceptional set E_δ (see Definition 1) associated with the system of linear forms $[L_j]$. Since $[L_j]$ correspond to the eigen-pairs $[\bar{\lambda}_j, \mathbf{u}_j]$ of $A^\dagger$ they are linearly independent linear forms with (real or complex) algebraic coefficients. Thus the subspace theorem asserts that E_δ is a finite collection of proper rational subspaces of $\mathbb{R}^d$. We denote by k_δ the number of subspaces forming E_δ.

Now we want to show that for all sufficiently large n there exist an integer $m \leqslant n$ such that $A^m \mathbf{z}_n$ does not lie in any element of E_δ. To this end we assume to the contrary that all $A^m \mathbf{z}_n$ lie in the subspaces forming E_δ and we divide the sequence of natural numbers $1,..., n$ into k_δ classes in such a way that two numbers m_1 and m_2 are in the same class if $A^{m_1} \mathbf{z}_n$ and $A^{m_2} \mathbf{z}_n$ lie in the same element of E_δ. Now let $n_*(k_\delta, d)$ be the number given in the van der Waerden theorem and let $n \geqslant n_*$. Then there exists an arithmetic progression $m_1,..., m_d$ in one of these subsequences. By Lemma 3 (d) the set of vectors $\{A^{m_1} \mathbf{z}_n, A^{m_2} \mathbf{z}_n,..., A^{m_d} \mathbf{z}_n\}$ forms a basis of the whole space $\mathbb{R}^d$, which contradicts the fact that they lie in one fixed rational proper subspace. Hence for any $\delta > 0$ and $n \geqslant n_*$ there exists $m_* \leqslant n$ such that $A^{m_*}\mathbf{z}_n$ does not lie in any element of E_δ.

Now, introducing the notation

$$\hat{\mathbf{z}}_n = A^{m_*}\mathbf{z}_n \tag{42}$$

one concludes from (41) that for any $\delta > 0$ and all $n \geqslant n_*$ the following equality and estimate hold

$$\sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{z}_n|^{2\alpha} = \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |\lambda_j|^{-2\alpha m_*} |L_j \hat{\mathbf{z}}_n|^{2\alpha} \tag{43}$$

$$\prod_{j=1}^{d} |L_j \hat{\mathbf{z}}_n| \geqslant \frac{1}{|\hat{\mathbf{z}}_n|^{\delta}}. \tag{44}$$

Inequality (44) may be rewritten as

$$\prod_{j=1}^{d} |L_j \hat{\mathbf{z}}_n| = \frac{1}{f(|\hat{\mathbf{z}}_n|)^{\delta}} \tag{45}$$

with some $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \leqslant r$, $\forall r > 0$.

Using (42) and (37) we obtain the existence of a constant $\lambda > 1$ such that

$$f(|\hat{\mathbf{z}}_n|) \leqslant |\hat{\mathbf{z}}_n| = |A^{m_*}\mathbf{z}_n| \leqslant \hat{\lambda}_{\max}^{m_*} |\mathbf{z}_n| \leqslant \hat{\lambda}_{\max}^{n} \sum_{j=1}^{d} \hat{\lambda}_j^{n} |L_j \mathbf{z}_n| \leqslant dD(\hat{\lambda}_{\max}\hat{\lambda}_{\text{geo}})^n \leqslant \lambda^n. \tag{46}$$

Note that $\prod_j \lambda_j = 1$. So, by (45) the quantities $B_{j,n} = (|\lambda_j|^{-m_*} f(|\hat{\mathbf{z}}_n|)^{\delta/d} |L_j \hat{\mathbf{z}}_n|)^{2\alpha}$, $j = 1,\ldots, d$ satisfy the constraint

$$\prod_{j=1}^{d} B_{j,n} = 1, \quad \forall n > n_*. \tag{47}$$

Thus applying (46) and the Lagrange multipliers minimization with the constraint (47) one gets

$$\sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |\lambda_j|^{-2\alpha m_*} |L_j \hat{\mathbf{z}}_n|^{2\alpha} = f(|\hat{\mathbf{z}}_n|)^{-2\alpha\delta/d} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} B_{j,n} \geqslant \lambda^{-2\alpha n\delta/d} \hat{\lambda}_{\text{geo}}^{2\alpha n} =: \hat{\lambda}_{\text{geo}}^{2\alpha n(1-\hat{\delta})}.$$

This and equality (43) yields the following lower bound for (33)

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{2\alpha n} |L_j \mathbf{k}|^{2\alpha} \geqslant \hat{\lambda}_{\text{geo}}^{2\alpha n(1-\hat{\delta})}. \tag{48}$$

3.2.2. Non-Diagonalizable Case

We move on to the general case where A is not irreducible (but, as assumed at the beginning of the proof, indecomposable over $\mathbb{Q}$). We denote by B the invariant irreducible sub-block of A given by its block upper triangular decomposition (26) and by S the rational invariant subspace associated with this block. We note that B as an irreducible matrix is diagonalizable.

Upper Bound. Note that

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant \min_{0 \neq \mathbf{k} \in S \cap \mathbb{Z}^d} \sum_{l=1}^{n} |B^l \mathbf{k}|^{2\alpha}. \tag{49}$$

The corresponding upper bound (38) for B is then also an upper bound for the whole matrix A. We note that geometric average of $\hat{\lambda}_j$ over S is equal to the geometric average of all $\hat{\lambda}_j$ associated with matrix A (i.e., over the whole space $\mathbb{R}^d$).

Lower Bound. According to our assumption A is indecomposable and thus the characteristic polynomial of A is of the form p^m for some irreducible p. All Jordan blocks of A have the same size m and different Jordan blocks correspond to distinct eigenvalues. Denote by b the number of the Jordan blocks in A and by λ_j, where $j = 1,..., l$ all these distinct eigenvalues . Since each λ_j has algebraic multiplicity m, we get $d = mb$. Let $\{\mathbf{v}_{j,h}\}_{j=1,...,b;\, h=0,...,m-1}$ be a basis (of $\mathbb{C}^d$) in which A admits the Jordan canonical form. As usually $L_{j,h}$ will denote the corresponding linear forms. Each $\mathbf{v}_{j,h}$ can be regarded as a generalized eigenvector of A associated with an eigenvalue λ_j. We assume that these generalized eigenvectors are ordered according to their degree, i.e., $\mathbf{v}_{j,h}$ satisfies the equation $(A - \lambda_j I)^{1+h} \mathbf{v}_{j,h} = 0$. Reordering the eigenvalues, if necessary, we can also assume that λ_1 has the largest modulus among all eigenvalues of A and hence $\hat{\lambda}_1 = |\lambda_1|$. Let $\mathbf{z}_n$ be the sequence of minimizers solving (21). We first note that for each n there exists $0 \leqslant h \leqslant m-1$ such that $L_{1,h}\mathbf{z}_n \neq 0$. Indeed, otherwise for all $h = 0,..., m-1$, $L_{1,h}\mathbf{z}_n = 0$ and consequently for any n and h $L_{1,h} A^n \mathbf{z}_n = 0$. The latter implies that the set of consecutive iterations $\{\mathbf{z}_n, A^1\mathbf{z}_n, A^2\mathbf{z}_n,...\}$ spans a proper rational A-invariant subspace of $\mathbb{R}^d$ which does not have any intersection with the subspace spanned by the generalized eigenvectors of A associated with eigenvalue λ_1. This clearly contradicts the irreducibility of p. Now, for given n we denote by $h(n)$ the biggest index h for which the condition $L_{1,h}\mathbf{z}_n \neq 0$ holds.

We have the following estimate

$$\hat{\lambda}_1^{2\alpha n} |L_{1,h(n)}\mathbf{z}_n|^{2\alpha} \leqslant \left(\sum_{j=1}^{b} \sum_{h=0}^{m-1} \left| \sum_{i=0}^{m-1-h} \lambda_j^{n-i} \binom{n}{i} L_{j,h+i}\mathbf{z}_n \right|^2 \right)^{\alpha} \tag{50}$$

$$\leqslant C_1 |A^n \mathbf{z}_n|^{2\alpha} \leqslant C_1 \sum_{l=1}^{n} |A^l \mathbf{z}_n|^{2\alpha} \leqslant C_2 n \hat{\lambda}_{\text{geo}}^{2\alpha n}, \tag{51}$$

where the last inequality follows from previously established upper bound.

From the Diophantine approximation and the assumption that $|L_{1,h(n)}\mathbf{z}_n| \neq 0$, there exists $\beta > 0$ such that (see ref. 24, p. 164)

$$|L_{1,h(n)}\mathbf{z}_n| \geqslant \frac{1}{|\mathbf{z}_n|^{\beta}}. \tag{52}$$

Thus combining (50) with (52) one gets

$$\hat{\lambda}_1^{2\alpha n} |\mathbf{z}_n|^{-2\alpha\beta} \leqslant \hat{\lambda}_1^{2\alpha n} |L_{1,h(n)}\mathbf{z}_n|^{2\alpha} \leqslant C_2 n \hat{\lambda}_{\text{geo}}^{2\alpha n}.$$

After rearrangements one obtains the following lower bound estimate for (21)

$$\frac{C}{n} \hat{\lambda}^{2\alpha n} \leqslant C |\mathbf{z}_n|^{2\alpha} \leqslant \sum_{l=1}^{n} |A^l \mathbf{z}_n|^{2\alpha} = \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}, \tag{53}$$

where

$$\hat{\lambda} = \left(\frac{\hat{\lambda}_1}{\hat{\lambda}_{\text{geo}}} \right)^{1/\beta}. \tag{54}$$

We note that ergodicity of A implies $\hat{\lambda}_1 > \hat{\lambda}_{\text{geo}} > 1$ (see (30), (19), and Proposition 5) which ensures non-triviality of this lower bound.

Now in order to finish the proof is suffices to combine the estimates (38), (48), and (53), and note that

$$\hat{\lambda}_{\text{geo}}^{2\alpha n} = e^{2\alpha \frac{h(A)}{d} n} = e^{2\alpha \hat{h}(A)\, n}$$

which yields (22). ∎

3.3. Proofs of Theorem 2 (ii), (iii), and Theorem 3

In this section we apply Theorem 4 to prove main theorems of Section 2.

In order to determine the dissipation time of $T_{\varepsilon,\alpha}$ one has to determine the asymptotics of $\|T^n_{\varepsilon,\alpha}\|$ when n goes to infinity. According to formulas (17) and (18) this problem reduces to problem (21) solved in previous sections.

Thus in view of Theorem (4) there exist constants C_1 and C_2 such that for any $\delta, \delta' > 0$ and sufficiently large n

$$C_1 e^{(1-\delta)\,2\alpha\tilde{h}(A)\,n} \leqslant \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant C_2 n e^{2\alpha\hat{h}(A)\,n} \leqslant C_2 e^{(1+\delta')\,2\alpha\hat{h}(A)\,n}.$$

Using formula (17)

$$e^{-\varepsilon C_2 e^{(1+\delta')\,2\alpha\hat{h}(A)\,n}} \leqslant \|T^n_{\varepsilon,\alpha}\| \leqslant e^{-\varepsilon C_1 e^{(1-\delta)\,2\alpha\tilde{h}(A)\,n}}.$$

Now when $n = n_{\text{diss}}$, we have

$$C_1 e^{(1-\delta)\,2\alpha\tilde{h}(A)\,n_{\text{diss}}} \leqslant \frac{1}{\varepsilon} \leqslant C_2 e^{(1+\delta')\,2\alpha\hat{h}(A)\,n_{\text{diss}}}$$

and

$$\frac{1}{(1+\delta')\,2\alpha\hat{h}(A)} (\ln(1/\varepsilon) - \ln C_2) \leqslant n_{\text{diss}} \leqslant \frac{1}{(1-\delta)\,2\alpha\tilde{h}(A)} (\ln(1/\varepsilon) - \ln C_1),$$

which proves part (ii) of Theorem 2, i.e., the logarithmic growth of dissipation time as a function of $1/\varepsilon$.

Moreover, using the definition of dissipation rate constant

$$R_{\text{diss}} = \lim_{\varepsilon \to 0} \frac{n_{\text{diss}}}{\ln(1/\varepsilon)}$$

we obtain

$$\frac{1}{(1+\delta')\,2\alpha\hat{h}(A)} \leqslant R_{\text{diss}} \leqslant \frac{1}{(1-\delta)\,2\alpha\tilde{h}(A)}.$$

Finally letting $\delta \to 0$ and $\delta' \to 0$ we arrive at the following results:

- The general case—Theorem 2 (iii)

$$\frac{d}{2\alpha\hat{h}(F)} \leqslant R_{\text{diss}} \leqslant \frac{1}{2\alpha\tilde{h}(F)}.$$

- The diagonalizable case—Theorem 3

$$R_{\text{diss}} = \frac{1}{2\alpha \hat{h}(F)}.$$

This completes the proof. ▮

4. DEGENERATE NOISE

In this section we compute the dissipation time for non-strictly contracting generalizations of α-stable transition operators. Instead of considering standard α-stable kernels of the form (2) one can allow for some degree of degeneracy of noise in chosen directions by introducing the following family of noise kernels

$$g_{\varepsilon,\alpha,B}(\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\varepsilon\,|B\mathbf{k}|^{2\alpha}}\mathbf{e}_{\mathbf{k}}(\mathbf{x}), \tag{55}$$

where B denotes any $d\times d$ matrix with $\det B = 0$.

We denote by $G_{\varepsilon,\alpha,B}$ the noise operator associated with $g_{\varepsilon,\alpha,B}$. The degeneracy of B immediately implies that $\|G_{\varepsilon,\alpha,B}\| = 1$ and hence the general considerations of Sections 1 and 2 do not apply here. The answer to the question whether or not the dissipation time is finite depends on the choice of matrix B.

For simplicity we concentrate on the case when B is diagonalizable.

We call the eigenvector of B nondegenerate if it corresponds to nonzero eigenvalue.

Theorem 5. Let F be any toral automorphism and $T_{\varepsilon,\alpha,B} = G_{\varepsilon,\alpha,B}U_F$. Assume that B is diagonalizable. Then

(i) If all nondegenerate eigenvectors of B^* lie in one proper invariant subspace of F then dissipation does not take place, i.e., $n_{\text{diss}} = \infty$.

(ii) Otherwise the following statements hold.

(a) $T_{\varepsilon,\alpha}$ has simple dissipation time iff F is not ergodic.

(b) $T_{\varepsilon,\alpha}$ has logarithmic dissipation time iff F is ergodic.

(c) If $T_{\varepsilon,\alpha,B}$ has logarithmic dissipation time then the dissipation rate constant satisfies the following bounds

$$\frac{1}{2\alpha\hat{h}(F)} \leqslant R_{\text{diss}} \leqslant \frac{1}{2\alpha\tilde{h}(F)},$$

with some constant $\tilde{h}(F) \leqslant \hat{h}(F)$. The equality is achieved for all diagonalizable automorphisms F.

Proof. We continue to use the convention $A = F^{\dagger}$. The general formula derived previously for $\|T^n_{\varepsilon, \alpha}\|$ (see (17)), will now take the form

$$\|T^n_{\varepsilon, \alpha, B}\| = \sup_{0 \neq \mathbf{k} \in \mathbb{Z}^d} e^{-\varepsilon \sum_{l=1}^{n} |BA^l \mathbf{k}|^{2\alpha}} = e^{-\varepsilon \inf_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |BA^l \mathbf{k}|^{2\alpha}}. \tag{56}$$

Thus we need to estimate

$$\inf_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |BA^l \mathbf{k}|^{2\alpha}.$$

To this end we denote by μ_j ($j = 1,..., d$) the eigenvalues of B and we construct a basis (of $\mathbb{C}^d$) $\{\mathbf{v}_j\}_{j=1,...,d}$ composed of normalized eigenvectors corresponding to eigenvalues μ_j. We denote by $P_{j_{j=1,...,d}}$ the set of eigenprojections on $\mathbf{v}_j$, and by L_j the set of corresponding linear forms, given by the eigenvectors $\mathbf{u}_j$ of $B^{\dagger}$, which are of course co-orthogonal to $\mathbf{v}_j$, i.e., $\langle \mathbf{u}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$. We have

$$\mathbf{x} = \sum_{j=1}^{d} P_j \mathbf{x} = \sum_{j=1}^{d} (L_j \mathbf{x})\, \mathbf{v}_j = \sum_{j=1}^{d} \langle \mathbf{x}, \mathbf{u}_j \rangle\, \mathbf{v}_j, \qquad \forall \mathbf{x} \in \mathbb{R}^d.$$

In subsequent computations the symbols C_1, C_2 denote some absolute constants values of which are subject to change during calculations.

We consider two cases.

(i) All nondegenerate eigenvectors of $B^{\dagger}$ lie in one proper subspace of F. We have the following estimates

$$|BA^l \mathbf{k}|^2 \geqslant C_1 \sum_{j=1}^{d} |P_j BA^l \mathbf{k}|^2 = C_1 \sum_{j=1}^{d} |\mu_j|^2\, |P_j A^l \mathbf{k}|^2$$
$$= C_1 \sum_{j=1}^{d} |\mu_j|^2\, |\langle A^l \mathbf{k}, \mathbf{u}_j \rangle|^2 = C_1 \sum_{j=1}^{d} |\mu_j|^2\, |\langle \mathbf{k}, F^l \mathbf{u}_j \rangle|^2$$

and

$$|BA^l \mathbf{k}|^2 \leqslant C_2 \sum_{j=1}^{d} |P_j BA^l \mathbf{k}|^2 = C_2 \sum_{j=1}^{d} |\mu_j|^2\, |P_j A^l \mathbf{k}|^2$$
$$= C_2 \sum_{j=1}^{d} |\mu_j|^2\, |\langle A^l \mathbf{k}, \mathbf{u}_j \rangle|^2 = C_2 \sum_{j=1}^{d} |\mu_j|^2\, |\langle \mathbf{k}, F^l \mathbf{u}_j \rangle|^2.$$

Since at least one of μ_j is zero and all nondegenerate vectors $\mathbf{u}_j$ lie in a proper invariant subspace of F, one easily sees that for each fixed n

$$\inf_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|BA^l\mathbf{k}|^{2\alpha}=\inf_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}\sum_{j=1}^{d}|\mu_j|^{2\alpha}\,|\langle\mathbf{k},F^l\mathbf{u}_j\rangle|^{2\alpha}=0.$$

(ii) In this case we have the following upper bound

$$\inf_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|BA^l\mathbf{k}|^{2\alpha}\leqslant\|B\|^{2\alpha}\inf_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\mathbf{k}|^{2\alpha}=\|B\|^{2\alpha}\min_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\mathbf{k}|^{2\alpha}. \quad (57)$$

In order to provide an appropriate lower bound we note that the set of vectors $\{F^h\mathbf{u}_j\}$, where $1\leqslant h\leqslant d$ and j runs through the indices of all nondegenerate eigenvectors of B, spans the whole space (otherwise all nondegenerate $\mathbf{u}_j$ would lie in one proper invariant subspace of F). We denote by $\{F^{h_i}\mathbf{u}_{j_i}\}$ $(1\leqslant i\leqslant d)$ a basis extracted from the above set. We can define now a new norm $|\cdot|_{\mathbf{u}}$ on $\mathbb{R}^d$ by

$$|\mathbf{x}|_{\mathbf{u}}^2=\sum_{i=1}^{d}|\langle\mathbf{x},F^{h_i}\mathbf{u}_{j_i}\rangle|^2$$

and compute

$$\begin{aligned}\sum_{l=1}^{dn}|BA^l\mathbf{k}|^{2\alpha}&=\sum_{l=0}^{n-1}\sum_{h=1}^{d}|BA^{dl+h}\mathbf{k}|^{2\alpha}\geqslant\sum_{l=0}^{n-1}\sum_{h=1}^{d}C_1\sum_{j=1}^{d}|P_jBA^{dl+h}\mathbf{k}|^{2\alpha}\\&=C_1\sum_{l=0}^{n-1}\sum_{h=1}^{d}\sum_{j=1}^{d}|\mu_j|^{2\alpha}\,|P_jA^{dl+h}\mathbf{k}|^{2\alpha}\geqslant C_1\sum_{l=0}^{n-1}\sum_{i=1}^{d}|\langle A^{dl+h_i}\mathbf{k},\mathbf{u}_{j_i}\rangle|^{2\alpha}\\&=C_1\sum_{l=0}^{n-1}\sum_{i=1}^{d}|\langle A^{dl}\mathbf{k},F^{h_i}\mathbf{u}_{j_i}\rangle|^{2\alpha}=C_1\sum_{l=0}^{n-1}|A^{dl}\mathbf{k}|_{\mathbf{u}}^{2\alpha}.\end{aligned}$$

Using the equivalence between norms $|\cdot|$ and $|\cdot|_{\mathbf{u}}$ and combining (57) with the above estimate we get

$$C_1\min_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=0}^{n-1}|A^{dl}\mathbf{k}|^{2\alpha}\leqslant\inf_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{dn}|BA^l\mathbf{k}|^{2\alpha}\leqslant\|B\|^{2\alpha}\min_{0\neq\mathbf{k}\in\mathbb{Z}^d}\sum_{l=1}^{dn}|A^l\mathbf{k}|^{2\alpha}.$$

This together with the obvious fact that $\hat{h}(A^d)=d\hat{h}(A)$ and the general estimate (22) reduces the proof back to the nondegenerate case considered in the previous section. ∎

5. TIME OF DECAY OF ε-COARSE-GRAINED STATES

The uncertainties in the initial preparation and the final measurement of the noiseless system give rise to non-cumulative random perturbations to the system. Alternatively, one can coarse-grain the initial and final states of the noiseless system by convoluting with the ε-noise kernel. That is, instead of the original operator $T_{\varepsilon,\alpha}$, we consider the operator $\hat{T}^n_{\varepsilon,\alpha}$ defined as

$$\hat{T}^n_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} U^n_F G_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} U_{F^n} G_{\varepsilon,\alpha} \tag{58}$$

and compute the number of iterations required to have the L^2-norm of the final state being e^{-1} times that of the initial state. We will show that for ergodic toral automorphisms the required number of iterations is essentially the same asymptotically as the dissipation time computed in the previous sections.

One can represent the action of U_F or more generally U^n_F in the Fourier series

$$U^n_F \mathbf{e}_{\mathbf{k}} = \sum_{0 \neq \mathbf{k}' \in \mathbb{Z}^d} u^{(n)}_{\mathbf{k},\mathbf{k}'} \mathbf{e}_{\mathbf{k}'},$$

where $u^{(1)}_{\mathbf{k},\mathbf{k}'}$ coincides with $u_{\mathbf{k},\mathbf{k}'}$ defined previously (cf. (13)) and

$$u^{(n)}_{\mathbf{k},\mathbf{k}'} = \sum_{0 \neq \mathbf{k}_1,\ldots,\mathbf{k}_{n-1} \in \mathbb{Z}^d} u_{\mathbf{k},\mathbf{k}_1} u_{\mathbf{k}_1,\mathbf{k}_2} \cdots u_{\mathbf{k}_{n-1},\mathbf{k}'}$$

which satisfies

$$\sum_{0 \neq \mathbf{k}' \in \mathbb{Z}^d} |u^{(n)}_{\mathbf{k},\mathbf{k}'}|^2 = 1, \quad \forall\, n, \mathbf{k}. \tag{59}$$

Then

$$\hat{T}^n_{\varepsilon,\alpha} \mathbf{e}_{\mathbf{k}_0} = G_{\varepsilon,\alpha} U^n_F G_{\varepsilon,\alpha} \mathbf{e}_{\mathbf{k}_0} = G_{\varepsilon,\alpha} U^n_F e^{-\varepsilon |\mathbf{k}_0|^2} \mathbf{e}_{\mathbf{k}_0} = e^{-\varepsilon |\mathbf{k}_0|^2} G_{\varepsilon,\alpha} \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} u^{(n)}_{\mathbf{k}_0,\mathbf{k}_n} \mathbf{e}_{\mathbf{k}_n}$$

$$= e^{-\varepsilon(|\mathbf{k}_0|^2 + |\mathbf{k}_n|^2)} \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} u^{(n)}_{\mathbf{k}_0,\mathbf{k}_n} \mathbf{e}_{\mathbf{k}_n}.$$

Now we define

$$S_n(\mathbf{k}_n) = \{\mathbf{k}_0 \in \mathbb{Z}^d \setminus \{0\} : u^{(n)}_{\mathbf{k}_0,\mathbf{k}_n} \neq 0\}.$$

Similar computations to these performed in Section 2 give the following general upper bound for $\|\hat{T}^n_{\varepsilon,\alpha}\|$

$$\|\hat{T}^n_{\varepsilon,\alpha} f\|^2 \leqslant \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \sum_{\mathbf{k}_0 \in S_n(\mathbf{k}_n)} |\hat{f}(\mathbf{k}_0)|^2 \sum_{\mathbf{k}_0 \in S_n(\mathbf{k}_n)} |u^{(n)}_{\mathbf{k}_0,\mathbf{k}_n}|^2. \tag{60}$$

For a toral automorphism one easily sees that

$$u^{(n)}_{\mathbf{k}_0,\mathbf{k}_n} = e^{-\varepsilon(|\mathbf{k}_0|^{2\alpha}+|A^n\mathbf{k}_0|^{2\alpha})}\delta_{\mathbf{k}_n, A^n\mathbf{k}_0} \tag{61}$$

and hence

$$\|\hat{T}^n_{\varepsilon,\alpha}\| = e^{-\varepsilon \min_{0\neq \mathbf{k}\in\mathbb{Z}^d}(|\mathbf{k}_0|^{2\alpha}+|A^l\mathbf{k}_0|^{2\alpha})}.$$

The arithmetic minimization problem (18) corresponding to the dissipation time of $\hat{T}_{\varepsilon,\alpha}$ now becomes

$$\min_{0\neq\mathbf{k}\in\mathbb{Z}^d}(|\mathbf{k}|^{2\alpha}+|A^n\mathbf{k}|^{2\alpha}). \tag{62}$$

The key observation is that, by the same arguments as before, similar estimates to these given in (32) hold

$$C_1\left(\sum_{j=1}^d \hat{\lambda}_j^{2n}\,|L_j\mathbf{k}|^2\right)^\alpha \leqslant |\mathbf{k}|^{2\alpha}+|A^n\mathbf{k}|^{2\alpha} \leqslant C_2\sum_{j=1}^d \hat{\lambda}_j^{2\alpha n}\,|L_j\mathbf{k}|^{2\alpha}. \tag{63}$$

The remaining computations are the same verbatim so the dissipation time of $T_{\varepsilon,\alpha}$ and $\hat{T}_{\varepsilon,\alpha}$ are equal asymptotically.

6. TIME SCALES IN KINEMATIC DYNAMO

In this section we briefly discuss the connection between the dissipation time and some characteristic time scales associated with kinematic dynamo, which concerns the generation of electromagnetic fields by mechanical motion. For a general setup and discussion we refer the reader to refs. 4 and 15 and references therein. Here we restrict ourselves only to necessary definitions.

Let $\mathbf{B}\in L^2_0(\mathbb{T}^d,\mathbb{R}^d)$ denote periodic, zero mean and divergence free magnetic field and let F be the time-1 map associated with the fluid velocity. We define the push-forward map

$$F_*\mathbf{B}(\mathbf{x}) = dF(F^{-1}(\mathbf{x}))\,\mathbf{B}(F^{-1}(\mathbf{x})).$$

The noisy push-forward map $P_{\varepsilon,\alpha}$ on $L^2_0(\mathbb{T}^d,\mathbb{R}^d)$ is then given by

$$P_{\varepsilon,\alpha}\mathbf{B} := G_{\varepsilon,\alpha}F_*\mathbf{B}, \tag{64}$$

where the convolution (the action of $G_{\varepsilon,\alpha}$) is applied component-wise.

It is said that the kinematic dynamo action (positive dynamo effect) occurs if the dynamo growth rate is positive, i.e.,

$$R_{\text{dyn}} = \lim_{n \to \infty} \frac{1}{n} \ln \|P^n_{\varepsilon,\alpha}\| > 0.$$

Moreover if

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln \|P^n_{\varepsilon,\alpha}\| > 0,$$

then the dynamo action is said to be fast; otherwise it is slow. The anti-dynamo action takes place if

$$\lim_{n \to \infty} \frac{1}{n} \ln \|P^n_{\varepsilon,\alpha}\| < 0.$$

Now we introduce the *threshold time* scale as

$$n_{th} = \max\{n\colon \|P^n_{\varepsilon,\alpha}\| > e \text{ such that } \|P^{n-1}_{\varepsilon,\alpha}\| \text{ or } \|P^{n+1}_{\varepsilon,\alpha}\| \leqslant e\}.$$

The threshold time $n_{th}(\varepsilon)$ is of order $O(1)$ as $\varepsilon \to 0$ for all fast kinematic dynamo systems. In the case of anti-dynamo action, $n_{th}(\varepsilon)$ captures the longest time scale on which the generation of the magnetic field still takes place. Finally $n_{th}(\varepsilon)$ is not defined for systems with do not exhibit any growth of magnetic field throughout the evolution. In the case of anti-dynamo we consider the time scale on which the generation of the magnetic field achieves its maximal value

$$n_p = \min\{n\colon \|P^n_{\varepsilon,\alpha}\| = \sup_m \|P^m_{\varepsilon,\alpha}\|\}$$

which is called the *peak time* of the anti-dynamo action.

Our next theorem establishes the relation between n_p, n_{th}, and n_{diss} for toral automorphisms. Thus $dF = F$ and

$$P_{\varepsilon,\alpha}\mathbf{B} = g_{\varepsilon,\alpha} * F(\mathbf{B} \circ F^{-1}).$$

Theorem 6. Let F be any toral automorphism. Then

(i) If F is nonergodic and has positive entropy then for all $0 < \varepsilon < R_{\text{diss}} \ln \rho_F$ the fast dynamo action takes place with dynamo growth rate satisfying

$$R_{\text{dyn}} = \ln \rho_F - \varepsilon R^{-1}_{\text{diss}} \xrightarrow{\varepsilon \to 0} \ln \rho_F > 0,$$

where ρ_F denotes the spectral radius of F. The threshold time n_{th} is of order $O(1)$ and if F is diagonalizable then $n_{th} \approx [R_{\text{dyn}}^{-1}]$.

(ii) If F is nonergodic and has zero entropy then anti-dynamo action occurs and for nondiagonalizable F,

$$\begin{aligned} n_p &\sim \frac{n_{th}}{\ln(n_{th})} \\ &\sim n_{\text{diss}} \\ &\approx R_{\text{diss}} \frac{1}{\varepsilon}. \end{aligned}$$

Moreover there exists a constant $0 < \gamma \leqslant d$ such that $\|P^{n_p}_{\varepsilon,\alpha}\| \sim (1/\varepsilon)^{\gamma}$. If F is diagonalizable then $\|P^{n}_{\varepsilon,\alpha}\|$ is strictly decreasing (in n) and, hence, $n_p = 0$ and n_{th} is not defined.

(iii) If F is ergodic then anti-dynamo action occurs and

$$n_p \approx n_{\text{diss}}.$$

In particular if F is diagonalizable then

$$\begin{aligned} n_p &\approx n_{th} - R_{\text{diss}} \ln(n_{th}) \\ &\approx R_{\text{diss}} \ln(1/\varepsilon) \\ &= \frac{1}{2\alpha \hat{h}(F)} \ln(1/\varepsilon) \end{aligned}$$

and

$$\|P^{n_p}_{\varepsilon,\alpha}\| \sim (1/\varepsilon)^{\frac{\ln \rho_F}{2\alpha \hat{h}(F)}}. \tag{65}$$

We see that even in the case of anti-dynamo action the magnetic field can still grow to relatively large magnitude when the noise is small (power-law in $1/\varepsilon$).

Proof. Representing the initial magnetic field $\mathbf{B} = (\mathbf{b}_1, ..., \mathbf{b}_d)$ in Fourier basis

$$\mathbf{B} = \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \hat{\mathbf{B}}(\mathbf{k})\, e_{\mathbf{k}}$$

one obtains

$$P_{\varepsilon,\alpha} \mathbf{B} = \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} F \hat{\mathbf{B}}(\mathbf{k})\, e^{-\varepsilon |A\mathbf{k}|^{2\alpha}} e_{A\mathbf{k}},$$

where we set $A = (F^{-1})^{\dagger}$. After n iterations

$$P^n_{\varepsilon,\alpha}\mathbf{B} = \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} F^n \hat{\mathbf{B}}(\mathbf{k})\, e^{-\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}}\, e_{A^n \mathbf{k}}.$$

Thus

$$\begin{aligned}
\|P^n_{\varepsilon,\alpha}\mathbf{B}\|^2 &\leqslant \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} |F^n \hat{\mathbf{B}}(\mathbf{k})|^2\, e^{-2\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}} \\
&\leqslant \max_{0 \neq \mathbf{k} \in \mathbb{Z}^d} e^{-2\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}} \sum_{0 \neq \mathbf{k} \in \mathbb{Z}^d} |F^n \hat{\mathbf{B}}(\mathbf{k})|^2 \\
&= e^{-2\varepsilon \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}}\, |F^n \mathbf{B}|^2 = e^{-2\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}_n|^{2\alpha}}\, |F^n \mathbf{B}|^2 \\
&\leqslant e^{-2\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}_n|^{2\alpha}}\, \|F^n\|^2\, |\mathbf{B}|^2,
\end{aligned}$$

where $\mathbf{k}_n$ denotes a solution of the minimization problem

$$\min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}.$$

The above calculation provides the following upper bound

$$\|P^n_{\varepsilon,\alpha}\| \leqslant e^{-\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}_n|^{2\alpha}}\, \|F^n\|.$$

On the other hand let $\mathbf{v}_n$ denote a unit vector satisfying $\|F^n\| = |F^n \mathbf{v}|$. One immediately sees that the above upper bound for $\|P^n_{\varepsilon,\alpha}\|$ is achieved for magnetic field of the form $\mathbf{B} = \mathbf{v}_n e_{\mathbf{k}_n}$. Thus

$$\|P^n_{\varepsilon,\alpha}\| = e^{-\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}_n|^{2\alpha}}\, \|F^n\|. \tag{66}$$

Now we consider the cases mentioned in the statement of the theorem

(i) Nonergodic, nonzero entropy case.
For any nonergodic map we have

$$\sum_{l=1}^{n} |A^l \mathbf{k}_n|^{2\alpha} \approx R^{-1}_{\text{diss}} n.$$

This implies the following asymptotics

$$\|P^n_{\varepsilon,\alpha}\| \approx e^{-\varepsilon R^{-1}_{\text{diss}} n}\, \|F^n\| \approx e^{(-\varepsilon R^{-1}_{\text{diss}} + \ln \rho_F)\, n + c_1 \ln n + c_2}, \tag{67}$$

where $c_1, c_2 \geqslant 0$ are constants (both equal 0 iff F is diagonalizable). Thus for $\varepsilon < R_{\rm diss} \ln \rho_F$ we have

$$R_{\rm dyn} = \ln \rho_F - \varepsilon R_{\rm diss}^{-1} \xrightarrow{\varepsilon \to 0} \ln \rho_F > 0.$$

The threshold time is clearly of order $O(1)$ and in diagonalizable case can be written as

$$n_{th} \approx \frac{1}{\ln \rho_F - \varepsilon R_{\rm diss}^{-1}} \xrightarrow{\varepsilon \to 0} \frac{1}{\ln \rho_F}.$$

(ii) Nonergodic, zero entropy case.
In this case $\ln \rho_F = 0$. Thus if F is nondiagonalizable then (67) reads

$$\|P^n_{\varepsilon,\alpha}\| \approx e^{-\varepsilon R_{\rm diss}^{-1} n} \|F^n\| \approx e^{-\varepsilon R_{\rm diss}^{-1} n + c_1 \ln n + c_2},$$

with $0 < c_1 \leqslant d$. This immediately yields

$$n_p \approx R_{\rm diss} \frac{c_1}{\varepsilon}, \qquad \frac{n_{th}}{\ln(n_{th})} \sim \frac{1}{\varepsilon}.$$

And moreover $\|P^{n_p}_{\varepsilon,\alpha}\| \sim (1/\varepsilon)^{c_1}$.

If F is diagonalizable then $\|F^n\| = 1$ and in this case $\|P^n_{\varepsilon,\alpha}\| \approx e^{-\varepsilon R_{\rm diss}^{-1} n}$ which implies $n_p = 0$.

(iii) If F is diagonalizable, then form (22) we know that for any $0 < \delta < 1$ and sufficiently large n

$$\lambda^n_{-\delta} = e^{(1-\delta)\, 2\alpha \hat{h}(A)\, n} \leqslant \min_{0 \neq \mathbf{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha} \leqslant e^{(1+\delta)\, 2\alpha \hat{h}(A)\, n} = \lambda^n_{+\delta}. \tag{68}$$

Thus for large n we have

$$\max_n e^{-\varepsilon \lambda^n_{+\delta}} \rho_F^n \leqslant \max_n \|P^n_{\varepsilon,\alpha}\| \leqslant \max_n e^{-\varepsilon \lambda^n_{-\delta}} \rho_F^n.$$

We obtain the following constraints for n_p

$$\frac{1}{\ln \lambda_{+\delta}} \ln\left(\frac{\ln \rho_F}{\ln \lambda_{+\delta}}\right) + \frac{1}{\ln \lambda_{-\delta}} \ln\left(\frac{1}{\varepsilon}\right) \leqslant n_p \leqslant \frac{1}{\ln \lambda_{-\delta}} \ln\left(\frac{\ln \rho_F}{\ln \lambda_{-\delta}}\right) + \frac{1}{\ln \lambda_{-\delta}} \ln\left(\frac{1}{\varepsilon}\right).$$

This gives

$$\frac{1}{\ln \lambda_{+\delta}} \leqslant \lim_{\varepsilon \to 0} \frac{n_p}{\ln(1/\varepsilon)} \leqslant \frac{1}{\ln \lambda_{-\delta}}.$$

Now since $\lambda_{\pm\delta} \to e^{2\alpha\hat{h}(F)}$ for $\delta \to 0$ the above estimation yields the following asymptotics

$$n_p \approx \frac{1}{2\alpha\hat{h}(F)} \ln(1/\varepsilon) \approx n_{\text{diss}}, \qquad n_{th} - R_{\text{diss}} \ln(n_{th}) \approx n_{\text{diss}}.$$

Similar asymptotic estimates (except for the constant) hold for nondiagonalizable F. ▮

APPENDIX A: AFFINE TRANSFORMATIONS

In this appendix we present a slight generalization of the results obtained in the paper. We consider here general affine transformations of the torus. The term *affine transformations* will be used here to refer to homeomorphisms of the torus with zero periodic but not necessary zero constant part (cf. Section 2.3), i.e., transformations of the form $\tilde{F} = F + \mathbf{c}$, where F is a toral automorphism and $\mathbf{c}$ is a constant shift vector.

We begin with a short discussion of the ergodicity of affine transforms.

The relation between ergodicity of a given affine transform $\tilde{F}$ and associated with it toral automorphism F is summarized in the following proposition (for the proof we refer to Appendix B)

Proposition 8. Let F be any toral automorphism. Then

(i) If F is ergodic then $\tilde{F}$ is also ergodic.

(ii) If F is not ergodic then $\tilde{F}$ is ergodic iff 1 is the only root of unity in the spectrum of F and $\mathbf{c} \cdot \mathbf{k} \notin \mathbb{Z}^d$ for any integer eigenvector $\mathbf{k}$ of $F^\dagger$.

Proof.

(i) Assume F is ergodic and for some $\mathbf{c}$, $\tilde{F} = F + \mathbf{c}$ is not ergodic. Then there exists non-constant $f \in L_0^2(\mathbb{T}^d)$ satisfying $f = f \circ \tilde{F}$ or in the Fourier representation

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k})\, \mathbf{e}_\mathbf{k} = \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{2\pi i A^{-1}\mathbf{k} \cdot \mathbf{c}} \hat{f}(A^{-1}\mathbf{k})\, \mathbf{e}_\mathbf{k} \tag{69}$$

where $A = F^\dagger$. Comparing the absolute values of the coefficients we get

$$|\hat{f}(\mathbf{k})| = |\hat{f}(A^{-n}\mathbf{k})| \tag{70}$$

for any integer n and any $\mathbf{k}$. However, ergodicity of F implies that $A^{-n}\mathbf{k} \neq \mathbf{k}$ for all $\mathbf{k} \neq 0$, which contradicts our assumption that $f \in L_0^2(\mathbb{T}^d)$.

(ii) We will use the following fact, which can be proved by simple application of rational canonical decomposition. For any $A \in SL(d, \mathbb{Z})$ the following conditions are equivalent

(a) A possesses in its spectrum a root of unity not equal to one.

(b) There exists nonzero $\mathbf{k} \in \mathbb{Z}^d$ and a positive integer n such that $\mathbf{k} + A\mathbf{k} + \cdots + A^{n-1}\mathbf{k} = 0$.

Now assume that 1 is the only root of unity in spectrum of F (and hence of A) and $\mathbf{c} \cdot \mathbf{k} \notin \mathbb{Z}^d$ for any integer eigenvector $\mathbf{k}$ of A, and that both F and $\tilde{F}$ are not ergodic. The latter assumption implies the existence of a non-constant $f \in L_0^2(\mathbb{T}^d)$ satisfying equations (69) and (70). Relation (70) clearly implies that if $\hat{f}(\mathbf{k}) \neq 0$ then $A^n\mathbf{k} = \mathbf{k}$ for some n. Moreover, since 1 is the only root of unity in spectrum of A, we have, in view of (b) that $A\mathbf{k} = \mathbf{k}$. Thus the only possible non-constant invariant functions of $\tilde{F}$ are single Fourier modes $\mathbf{e}_\mathbf{k}$ corresponding to integer eigenvectors of A. But if such a Fourier mode is invariant under $\tilde{F}$ then directly form (A1) one concludes that $e^{2\pi i \mathbf{k} \cdot \mathbf{c}} = 1$ or equivalently $\mathbf{k} \cdot \mathbf{c} \in \mathbb{Z}^d$, for some integer eigenvector of A. To prove the converse we assume that F is not ergodic and consider two cases:

Case 1. A possesses in its spectrum a root of unity not equal to one. In this case according to condition (b) there exists nonzero $\mathbf{k} \in \mathbb{Z}^d$ and a positive integer n such that $\mathbf{k} + A\mathbf{k} + \cdots + A^{n-1}\mathbf{k} = 0$, which implies in particular that $A^n\mathbf{k} = \mathbf{k}$ and $A\mathbf{k} \neq \mathbf{k}$. Now we define the function

$$f = \mathbf{e}_\mathbf{k} + e^{2\pi i \mathbf{k} \cdot \mathbf{c}} \mathbf{e}_{A\mathbf{k}} + \cdots + e^{2\pi i (\sum_{l=0}^{n-2} A^l \mathbf{k}) \cdot \mathbf{c}} \mathbf{e}_{A^{n-1}\mathbf{k}}$$

which clearly satisfies the condition $f = f \circ \tilde{F}$. This proves that $\tilde{F}$ is not ergodic.

Case 2. There exists integer eigenvector of A such that $\mathbf{k} \cdot \mathbf{c} \in \mathbb{Z}^d$. Then clearly for such $\mathbf{k}$, $f = e_\mathbf{k}$ is $\tilde{F}$-invariant and hence again $\tilde{F}$ is not ergodic. ∎

We recall that $\mathbf{c} = (c_1, ..., c_d)$ generates ergodic shift on the torus iff $1, c_1, ..., c_d$ are linearly independent over rationals. Thus as a direct consequence of the above proposition we get

Corollary 5. If F is not ergodic and 1 is the only root of unity in the spectrum of F then $\tilde{F}$ is ergodic for all vectors $\mathbf{c}$ generating ergodic shifts on the torus.

Now we are in a position to state and prove the generalization of Theorem 2 from Section 2.3 to the case of affine transforms (the corresponding generalizations of Theorem 3 and Corollary 3 are straightforward).

Theorem 7. Let $\tilde{F}$ be any affine transformation on the torus $\mathbb{T}^d$, F associated with $\tilde{F}$ toral automorphism and $T_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} U_{\tilde{F}}$. Then

(i) $T_{\varepsilon,\alpha}$ has simple dissipation time iff F is not ergodic.

(ii) $T_{\varepsilon,\alpha}$ has logarithmic dissipation time iff F is ergodic.

(iii) If $T_{\varepsilon,\alpha}$ has logarithmic dissipation time then the dissipation rate constant satisfies the following constraint

$$\frac{1}{2\alpha \hat{h}(F)} \leqslant R_{\text{diss}} \leqslant \frac{1}{2\alpha \tilde{h}(F)},$$

where $\tilde{h}(F) \leqslant \hat{h}(F)$ is certain positive constant.

Remark 1. The dissipation time of an affine transformation $\tilde{F}$ is determined by ergodic properties of its linear part F and hence not by ergodic properties of $\tilde{F}$ itself. In particular all ergodic affine transformations associated with nonergodic toral automorphisms (cf. Proposition 8) have simple dissipation time.

Proof of Theorem 7. Specializing the general calculations of dissipation time presented in Section 2.2 to the case of affine transformations $\tilde{F} = F + \mathbf{c}$, with nonzero $\mathbf{c}$, one easily finds the following counterparts of formulas (16) and (17)

$$u_{\mathbf{k},\mathbf{k}'} = e^{2\pi i \mathbf{k}\cdot\mathbf{c}} \delta_{A\mathbf{k},\mathbf{k}'},$$

$$\mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n) = e^{2\pi i (\sum_{l=0}^{n-1} A^l \mathbf{k})\cdot\mathbf{c}} e^{-\varepsilon \sum_{l=1}^{n} |A^l \mathbf{k}|^{2\alpha}} \delta_{A^n \mathbf{k}_0, \mathbf{k}_n}.$$

Now, in order to determine the dissipation time of $T_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} U_{\tilde{F}}$ one has to determine the asymptotics of $\|T^n_{\varepsilon,\alpha}\|$ as n goes to infinity. According to the above formulas and formulas (15) and (17) from Section 2.2 the value of $\|T^n_{\varepsilon,\alpha}\|$ does not depend on $\mathbf{c}$, which reduces the proof to the case $\mathbf{c} = 0$ considered in the main body of the paper. ∎

APPENDIX B: PROOFS OF SOME ELEMENTARY FACTS

Proof of Proposition 2. The proof will be based on the Riesz convexity theorem (see ref. 27, pp. 93–100) which states that for any operator

T defined on $L^p(\mathbb{T}^d)$, $1 \leqslant p \leqslant \infty$, $\ln \|T\|_p$ is a convex function of p^{-1}. On the space $L^p(\mathbb{T}^d)$ we consider the operator $\tilde{T} := T_{\varepsilon,\alpha} - \langle \cdot \rangle$ and we have the relation $\tilde{T}^n f = T^n_{\varepsilon,\alpha}(f - \langle f \rangle)$, $\forall f \in L^p(\mathbb{T}^d)$, $n \geqslant 1$ because $T_{\varepsilon,\alpha}$ is conservative. Now since $\|f - \langle f \rangle\|_p \leqslant 2\,\|f\|_p$, it follows that

$$\|\tilde{T}^n\|_p \leqslant 2\,\|T^n_{\varepsilon,\alpha}\|_{p,0} \leqslant 2 \tag{71}$$

$$\|T^n_{\varepsilon,\alpha}\|_{p,0} \leqslant \|\tilde{T}^n\|_p \tag{72}$$

for $1 \leqslant p \leqslant \infty$, $n \geqslant 1$. The Riesz convexity theorem implies that if $p < q < \infty$

$$\ln \|\tilde{T}^n\|_q \leqslant \frac{p}{q} \ln \|\tilde{T}^n\|_p + \left(1 - \frac{p}{q}\right) \ln \|\tilde{T}^n\|_\infty \tag{73}$$

while if $1 < q < p$

$$\ln \|\tilde{T}^n\|_q \leqslant \left(\frac{1 - 1/q}{1 - 1/p}\right) \ln \|\tilde{T}^n\|_p + \left(1 - \frac{1 - 1/q}{1 - 1/p}\right) \ln \|\tilde{T}^n\|_1. \tag{74}$$

From (73)–(74) we have the interpolation relations

$$\|\tilde{T}^n\|_q \leqslant \|\tilde{T}^n\|_p^{p/q}\, \|\tilde{T}^n\|_\infty^{1-p/q}, \qquad p < q < \infty \tag{75}$$

$$\|\tilde{T}^n\|_q \leqslant \|\tilde{T}^n\|_p^{(1-q^{-1})/(1-p^{-1})}\, \|\tilde{T}^n\|_1^{1-(1-q^{-1})/(1-p^{-1})}, \qquad 1 < q < p \tag{76}$$

which, along with (71)–(72), imply

$$\|T^n_{\varepsilon,\alpha}\|_{q,0} \leqslant 2\,\|T^n_{\varepsilon,\alpha}\|_{p,0}^{p/q}, \qquad p < q < \infty$$

$$\|T^n_{\varepsilon,\alpha}\|_{q,0} \leqslant 2\,\|T^n_{\varepsilon,\alpha}\|_{p,0}^{(1-q^{-1})/(1-p^{-1})}, \qquad 1 < q < p.$$

This proves that the order of divergence of $n_{\text{diss}}(p)$ are the same for $1 < p < \infty$. Estimates (B5)–(B6) also show that the order of divergence of $n_{\text{diss}}(1)$ and $n_{\text{diss}}(\infty)$ is at least as high as $n_{\text{diss}}(p)$, $1 < p < \infty$. ∎

Proof of Lemma 2. Using the notation introduced in Section 2.2 one has

$$T^n_{\varepsilon,\alpha}\mathbf{e}_{\mathbf{k}_0} = (G_{\varepsilon,\alpha}U)^n\,\mathbf{e}_{\mathbf{k}_0} = (G_{\varepsilon,\alpha}U)^{n-1} \sum_{0 \neq \mathbf{k}_1 \in \mathbb{Z}^d} u_{\mathbf{k}_0,\mathbf{k}_1} e^{-\varepsilon |\mathbf{k}_1|^{2\alpha}} \mathbf{e}_{\mathbf{k}_1}$$

$$= \sum_{0 \neq \mathbf{k}_1,\ldots,\mathbf{k}_n \in \mathbb{Z}^d} u_{\mathbf{k}_0,\mathbf{k}_1} u_{\mathbf{k}_1,\mathbf{k}_2} \cdots u_{\mathbf{k}_{n-1},\mathbf{k}_n} e^{-\varepsilon \sum_{l=1}^n |\mathbf{k}_l|^{2\alpha}} \mathbf{e}_{\mathbf{k}_n} = \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n)\,\mathbf{e}_{\mathbf{k}_n}.$$

We note that for any n and $\mathbf{k}_n \in \mathbb{Z}^d$, the sequence $\mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n)$ (indexed by $\mathbf{k}_0 \in \mathbb{Z}^d$) belongs to $l^2(\mathbb{Z}^d)$. Indeed, using the Cauchy–Schwarz inequality and identity (13) one gets for $n = 2$,

$$\sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} |\mathscr{U}_2(\mathbf{k}_0, \mathbf{k}_2)|^2$$

$$= \sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} \left| \sum_{0 \neq \mathbf{k}_1 \in \mathbb{Z}^d} u_{\mathbf{k}_0, \mathbf{k}_1} u_{\mathbf{k}_1, \mathbf{k}_2} e^{-\varepsilon(|\mathbf{k}_1|^{2\alpha} + |\mathbf{k}_2|^{2\alpha})} \right|^2$$

$$\leqslant \sum_{0 \neq \mathbf{k}_0 \in \mathbb{Z}^d} \sum_{0 \neq \mathbf{k}_1 \in \mathbb{Z}^d} |u_{\mathbf{k}_0, \mathbf{k}_1}|^2 e^{-\varepsilon |\mathbf{k}_1|^{2\alpha}} \sum_{0 \neq \mathbf{k}_1 \in \mathbb{Z}^d} |u_{\mathbf{k}_1, \mathbf{k}_2}|^2 e^{-\varepsilon |\mathbf{k}_1|^{2\alpha}} e^{-2\varepsilon |\mathbf{k}_2|^{2\alpha}}$$

$$\leqslant \sum_{0 \neq \mathbf{k}_1 \in \mathbb{Z}^d} e^{-\varepsilon |\mathbf{k}_1|^{2\alpha}} \sum_{0 \neq \mathbf{k}_1 \in \mathbb{Z}^d} e^{-\varepsilon |\mathbf{k}_1|^{2\alpha}} e^{-2\varepsilon |\mathbf{k}_2|^{2\alpha}} = K e^{-2\varepsilon |\mathbf{k}_2|^{2\alpha}},$$

where K denotes a constant depending only on ε and α. Similar estimates hold for $n > 2$.

Now applying the Cauchy–Schwarz inequality in (14) we get

$$\|T^n_{\varepsilon, \alpha} f\|^2 \leqslant \sum_{0 \neq \mathbf{k}_n \in \mathbb{Z}^d} \sum_{\mathbf{k}_0 \in \mathscr{S}_n(\mathbf{k}_n)} |\hat{f}(\mathbf{k}_0)|^2 \sum_{\mathbf{k}_0 \in \mathscr{S}_n(\mathbf{k}_n)} |\mathscr{U}_n(\mathbf{k}_0, \mathbf{k}_n)|^2. \quad \blacksquare \tag{77}$$

Proof of part (i) of Theorem 2. In view of Theorem 1 it suffices to construct an eigenfunction of U_F which belongs to $L^2_0(\mathbb{T}^d) \cap H^{2\alpha}(\mathbb{T}^d)$. Directly from Proposition 3 one concludes that F, and hence also A, possesses a root of unity in its spectrum. This means that $A^m \mathbf{k}_0 = \mathbf{k}_0$, for some m and certain nonzero vector $\mathbf{k}_0$, which can be chosen to be an integer. Now we define

$$f = \mathbf{e}_{\mathbf{k}_0} + \mathbf{e}_{A\mathbf{k}_0} + \cdots + \mathbf{e}_{A^{m-1}\mathbf{k}_0}.$$

Obviously $f \in L^2_0(\mathbb{T}^d) \cap H^{2\alpha}(\mathbb{T}^d)$, for any α. To complete the proof it suffices to notice that

$$U_F f = \mathbf{e}_{A\mathbf{k}_0} + \mathbf{e}_{A^2\mathbf{k}_0} + \cdots + \mathbf{e}_{A^m\mathbf{k}_0} = \mathbf{e}_{\mathbf{k}_0} + \mathbf{e}_{A\mathbf{k}_0} + \cdots + \mathbf{e}_{A^{m-1}\mathbf{k}_0} = f. \quad \blacksquare$$

Proof of Proposition 3. For the purposes of the proof we use the following abbreviation

- $PRS(\mathbb{R}^d)$—proper rational subspace of $\mathbb{R}^d$.
- $PIS(A, \mathbb{R}^d)$—proper A-invariant subspace of $\mathbb{R}^d$.
- $PRIS(A, \mathbb{R}^d)$—proper rational A-invariant subspace of $\mathbb{R}^d$.

(a) ⇒ (b) Suppose there exists $PRIS(A, \mathbb{R}^d)$ S_1. Let A_1 be a matrix representing invariant rational block associated with S_1. Then A_1 is rational matrix and its characteristic polynomial P_1 belongs to $\mathbb{Q}[x]$. Let P denote the characteristic polynomial of A. Then $P = P_1 P_2$ and since both $P, P_1 \in \mathbb{Q}[x]$ then also $P_2 \in \mathbb{Q}[x]$, which means P and hence A is not irreducible.

(b) ⇒ (c) Assume there exists $PRS(\mathbb{R}^d)$ S contained in $PIS(A, \mathbb{R}^d)$ V. Take any rational vector $\mathbf{q} \in S$ and let $d_0 = \dim V$ then the set $\{\mathbf{q}, A\mathbf{q}, ..., A^{d_0-1}\mathbf{q}\}$ spans $PRIS(A, \mathbb{R}^d)$.

(c) ⇒ (d) Assume that for given $\mathbf{q}$ and an arithmetic sequence $n_1, ..., n_d$, the set $S = \{A^{n_1}\mathbf{q}, A^{n_2}\mathbf{q}, ..., A^{n_d}\mathbf{q}\}$ does not form a basis. Since for some fixed integer r, $n_l = n_1 + (l-1)\, r$, we have $A^{n_l}\mathbf{q} = (A^r)^{l-1} A^{n_1}\mathbf{q} = (A^r)^{l-1}\, \hat{\mathbf{q}}$, where $\hat{\mathbf{q}} = A^{n_1}\mathbf{q}$. Now consider the biggest subset $S_0 = \{\hat{\mathbf{q}}, A^r\hat{\mathbf{q}}, (A^r)^2\, \hat{\mathbf{q}}, ..., (A^r)^{d_0-1}\, \hat{\mathbf{q}}\}$ such that $d_0 < d$ and S_0 is linearly independent. Obviously S_0 spans a $PRIS(A^r)$ which is also a $PRIS(A)$.

(d) ⇒ (a) Suppose that characteristic polynomial P of A is not irreducible in $\mathbb{Q}[x]$. Then $P = P_1 P_2$, with $P_1, P_2 \in \mathbb{Q}[x]$. From the Cayley–Hamilton theorem we get that $0 = P(A) = P_1(A)\, P_2(A)$. Hence for any nonzero rational vector $\mathbf{q}$, either (1) $P_2(A)\,\mathbf{q} = 0$ or (2) $\hat{\mathbf{q}} := P_2(A)\,\mathbf{q} \neq 0$ and $P_1(A)\,\hat{\mathbf{q}} = 0$. Since $\max\{\deg(P_1, P_2)\} < d$, there exists a nonzero rational vector $\tilde{\mathbf{q}}$ (namely $\mathbf{q}$ or $\hat{\mathbf{q}}$) such that the set of iterations $\{\tilde{\mathbf{q}}, A\tilde{\mathbf{q}}, A^2\tilde{\mathbf{q}}, ..., A^{d-1}\tilde{\mathbf{q}}\}$ does not form a basis of $\mathbb{R}^d$.

(e) ⇒ (f) Assume there exist nonzero $\mathbf{q} \in \mathbb{Q}^d$ orthogonal to certain $PIS(A, \mathbb{R}^d)$ V. Then for any n and any $f \in V$, $\langle (A^\dagger)^n\,\mathbf{q}, f\rangle = \langle \mathbf{q}, A^n f\rangle = 0$ and hence $S = \{\mathbf{q}, A^\dagger\mathbf{q}, (A^\dagger)^2\,\mathbf{q}, ..., (A^\dagger)^{d-1}\,\mathbf{q}\}$, cannot form a basis, which in view of equivalence (a) ⇔ (d) implies reducibility of $A^\dagger$.

(f) ⇒ (g) Suppose there exists $PIS(A, \mathbb{R}^d)$ V contained in certain $PRS(\mathbb{R}^d)$ S. Since S is rational, $S^\perp$ is also rational. Consider any rational vector $\mathbf{q} \in S^\perp$, then $\langle \mathbf{q}, f\rangle = 0$ for any $f \in V$.

(g) ⇒ (b) If there exists $PRIS(\mathbb{R}^d)$, then this subspace is A-invariant and contained in $PRS(\mathbb{R}^d)$, i.e., in itself.

Now since (b) is equivalent to (a) it is enough to establish the equivalence between (a) and (e) to complete the proof. But the latter equivalence is obvious in view of the fact that A and $A^\dagger$ have the same characteristic polynomial. ▮

Proof of Proposition 5. Suppose A is a toral automorphism of zero entropy. The latter property is equivalent to the fact that all the eigenvalues of A are of modulus 1. Let P_A be a characteristic polynomial of A. Consider any irreducible over $\mathbb{Z}$ factor P of polynomial P_A and construct a toral automorphism B such that its characteristic polynomial is equal to P. Obviously all the eigenvalues of B are also the eigenvalues of A, and each eigenvalue of A can be found among eigenvalues of some matrix B of this type. Irreducibility of P implies irreducibility and hence diagonalizability of B.

Thus for any nonzero vector $\mathbf{k} \in \mathbb{Z}^d$ and any positive integer n the following estimate holds $|B^n\mathbf{k}| \leqslant |\mathbf{k}|$, which implies the existence (for each $\mathbf{k}$) of some integer r such that $B^r\mathbf{k} = \mathbf{k}$.

The latter shows that all the eigenvalues of B (and hence also of A) are roots of unity. ∎

Proof of Proposition 6. We first show that irreducible polynomials $P \in \mathbb{Q}[x]$ do not have repeated roots. Indeed suppose λ is a root of P of multiplicity greater that 1, then λ is also a root of a derivative polynomial $P' \in \mathbb{Q}[x]$. Since the minimal polynomial of λ must divide both P and P' and $\deg(P') < \deg(P)$ one immediately concludes that P is not irreducible. Now, suppose $A \in GL(d, \mathbb{Q})$ is completely decomposable over $\mathbb{Q}$ and let (25) be its block diagonal decomposition into irreducible blocks. Each P_{A_j}, as a characteristic polynomial of A_j, is irreducible over $\mathbb{Q}$ and hence does not possesses repeated roots, which implies diagonalizability of each A_j and hence of A. On the other hand if A is diagonalizable then its minimal polynomial does not possesses repeated roots, which implies that all characteristic polynomials associated with elementary divisors are (first powers of) irreducible polynomials. This implies irreducibility of each block in representation (25). ∎

Proof of Proposition 7. Let P_A be the characteristic polynomial of an irreducible matrix $A \in GL(d, \mathbb{Q})$. Since P_A is an irreducible element of $\mathbb{Q}[x]$ it does not possesses repeated roots (see the proof of Proposition (6). ∎

ACKNOWLEDGMENTS

The research of AF is supported in part by the grant from U.S. National Science Foundation, DMS-9971322 and UC Davis Chancellor's Fellowship.

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